Discretization of Planar Curve Motions and Discrete Integrable Systems

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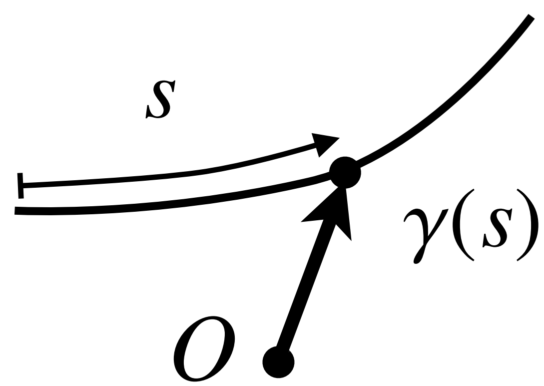
Introduction: motion of planar smooth curves and mKdV equation

Motion of Planar Curve and mKdV (I): Frenet Frame

- **Planar curve:** $$\gamma(s) = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} \in \mathbb{R}^2$$



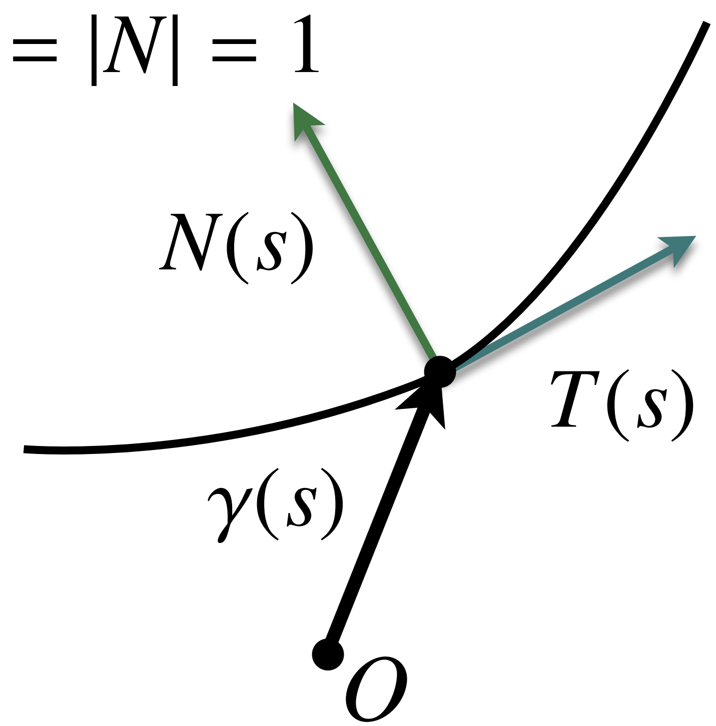
s : arc-length $$\sqrt{(dx)^2 + (dy)^2} = ds$$

$$\longleftrightarrow \quad |\gamma'| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = 1$$

- **Frenet frame:** $F(s) = [T(s), N(s)] \quad |T| = |N| = 1$

$$T(s) := \gamma'(s) = \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix}$$

$$N(s) := R\left(\frac{\pi}{2}\right)\gamma'(s) = \begin{bmatrix} -y'(s) \\ x'(s) \end{bmatrix}$$

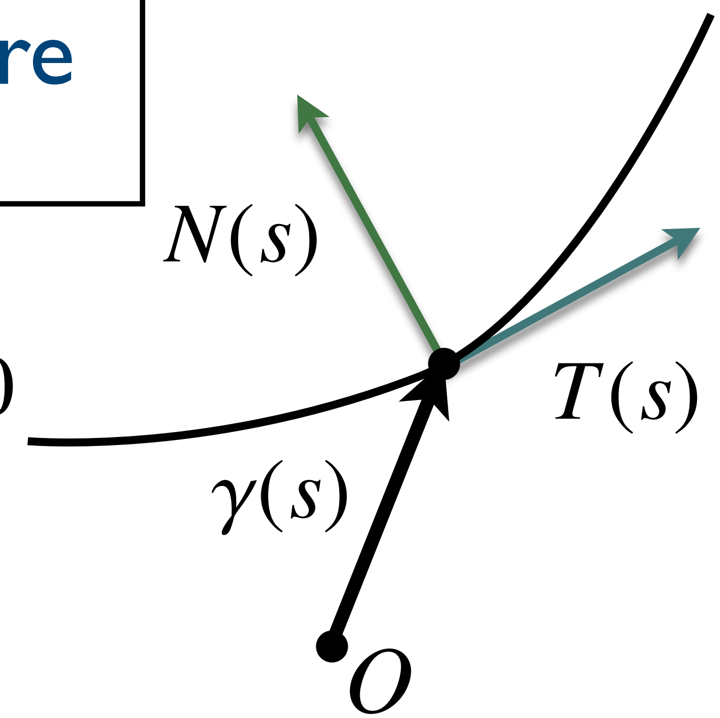


Motion of Planar Curve and mKdV (2): Frenet Formula

Frenet formula:

$$\frac{d}{ds}F(s) = F(s)\begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}$$

κ: curvature



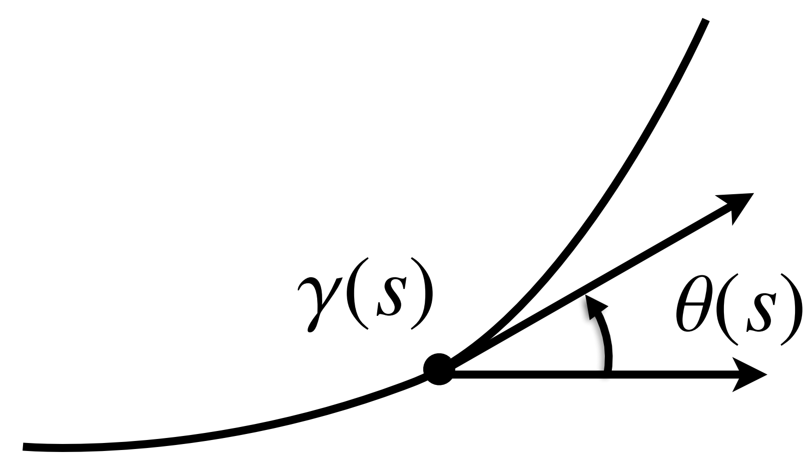
$$|T|^2 = (T,T) = 1 \to (T',T) + (T,T') = 2(T,T') = 0$$

$$\to T' = \kappa N, \quad N' = -\kappa T \quad \text{for } {}^{\exists}\kappa(s)$$

Potential function:

$$T = \gamma' = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$\theta(s)$: "turning angle" (potential function)



$$T' = \gamma'' = \theta' \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \theta' N \to \boxed{\theta' = \kappa}$$

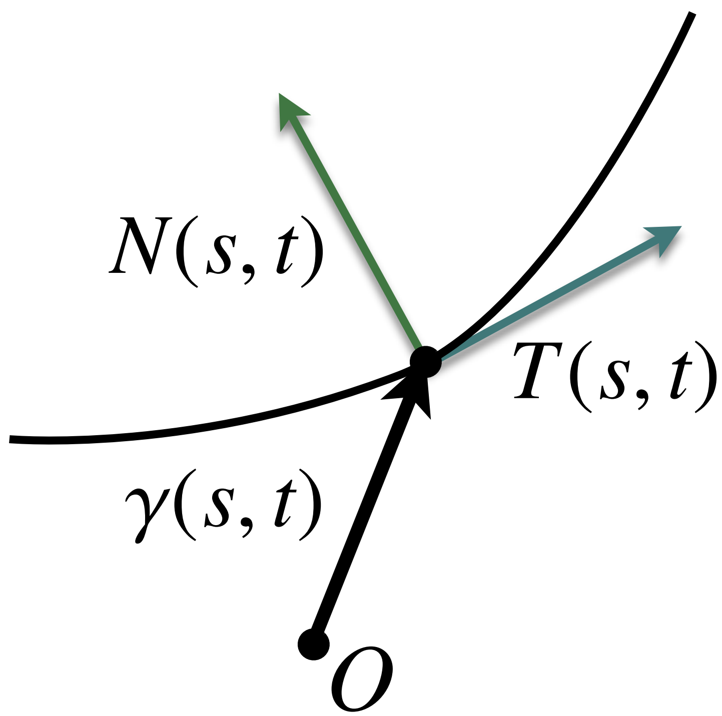
Motion of Planar Curve and mKdV (3): Curve motion

Isoperimetric motion:

t : time (deformation parameter) $\quad \gamma = \gamma(s,t)$ etc.

Requirement: $|\gamma'| = 1$ for all t (isoperimetric condition, 等周条件)

$$\longrightarrow \quad \frac{\partial}{\partial t}|\gamma'|^2 = 2(\gamma', \gamma_t') = 0$$



$$\gamma_t = f(s,t)T(s,t) + g(s,t)N(s,t)$$

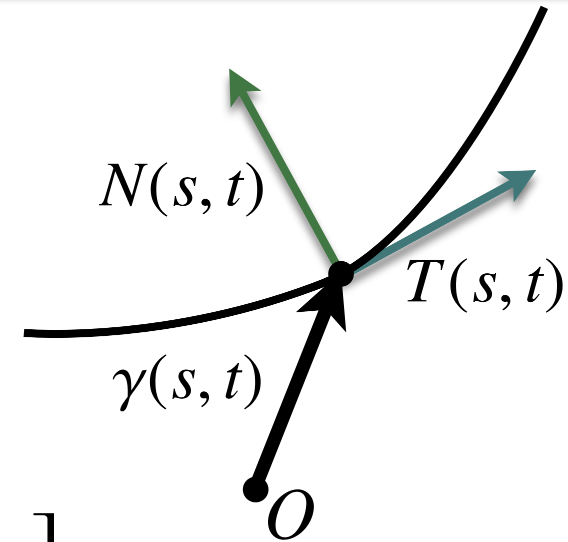
$$\to \gamma_t' = (f' - g\kappa)T + (g' + f\kappa)N$$

$$\to f' = g\kappa, \quad T_t = (g' + f\kappa)N, \quad N_t = -(g' + f\kappa)T$$

$$\frac{\partial}{\partial t}F(x,t) = F(x,t)\begin{bmatrix} 0 & -(g' + f\kappa) \\ g' + f\kappa & 0 \end{bmatrix}$$

Motion of Planar Curve and mKdV (4): Curve motion

Isoperimetric motion of planar curve:



$$F = [T, N], \quad \frac{\partial}{\partial s}F = FU, \quad \frac{\partial}{\partial t}F = FV$$

$$U(s,t) = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}, \quad V(s,t) = \begin{bmatrix} 0 & -(g' + f\kappa) \\ g' + f\kappa & 0 \end{bmatrix} \qquad f' = g\kappa$$

- **Compatibility condition:** $F_{st} = F_{ts}$

$$FVU + FU_t = FUV + FV_t \;\to\; U_t - V_s = [U, V] \;\to\; \boxed{\kappa_t = g_{ss} + g\kappa^2 + f\kappa_s}$$

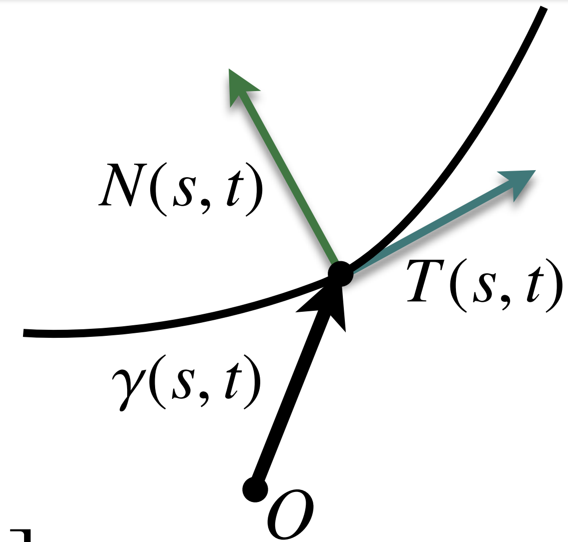
In particular, choose: $g = -\kappa_s, \quad f = -\dfrac{\kappa^2}{2}$

(potential) modified KdV equation

$$\kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0 \qquad \text{or} \qquad \theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0 \qquad \kappa = \theta'$$

Motion of Planar Curve and mKdV (5): Summary

Isoperimetric motion of planar curve described by mKdV eq.



$$F = [T, N], \quad \frac{\partial}{\partial s}F = FU, \quad \frac{\partial}{\partial t}F = FV$$

$$U(s,t) = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}, \quad V(s,t) = \begin{bmatrix} 0 & \kappa'' + \frac{\kappa^3}{2} \\ -\kappa'' - \frac{\kappa^3}{2} & 0 \end{bmatrix}$$

$$\kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0 \quad \text{or} \quad \theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0 \qquad \kappa = \theta'$$

cf. mKdV hierarchy

$$f = -\Omega^{n-1}\kappa_x, \quad g = -\partial_x^{-1}\left(\kappa\Omega^{n-1}\kappa_x\right), \quad \Omega = \partial_x^2 + \kappa^2 + \kappa_x\partial_x^{-1}\kappa$$

$$\longrightarrow \quad \kappa_t = -\Omega^n\kappa_x, \quad n = 1, 2, 3, \ldots$$

G. L. Lamb Jr., Phys. Rev. Lett. **37**(1976) 235-237

R.E. Goldstein, D.M. Petrich, Phys. Rev. Lett. **67**(1991) 3203-3206

井ノ口順一, 「曲線とソリトン」, 朝倉書店(2010)

mKdV and Curve Motion: Exact solutions

N-soliton solution to mKdV equation:

$$\tau = \begin{vmatrix} f_0^{(1)} & f_1^{(1)} & \cdots & f_{N-1}^{(1)} \\ f_0^{(2)} & f_1^{(2)} & \cdots & f_{N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_0^{(N)} & f_1^{(N)} & \cdots & f_{N-1}^{(N)} \end{vmatrix}$$

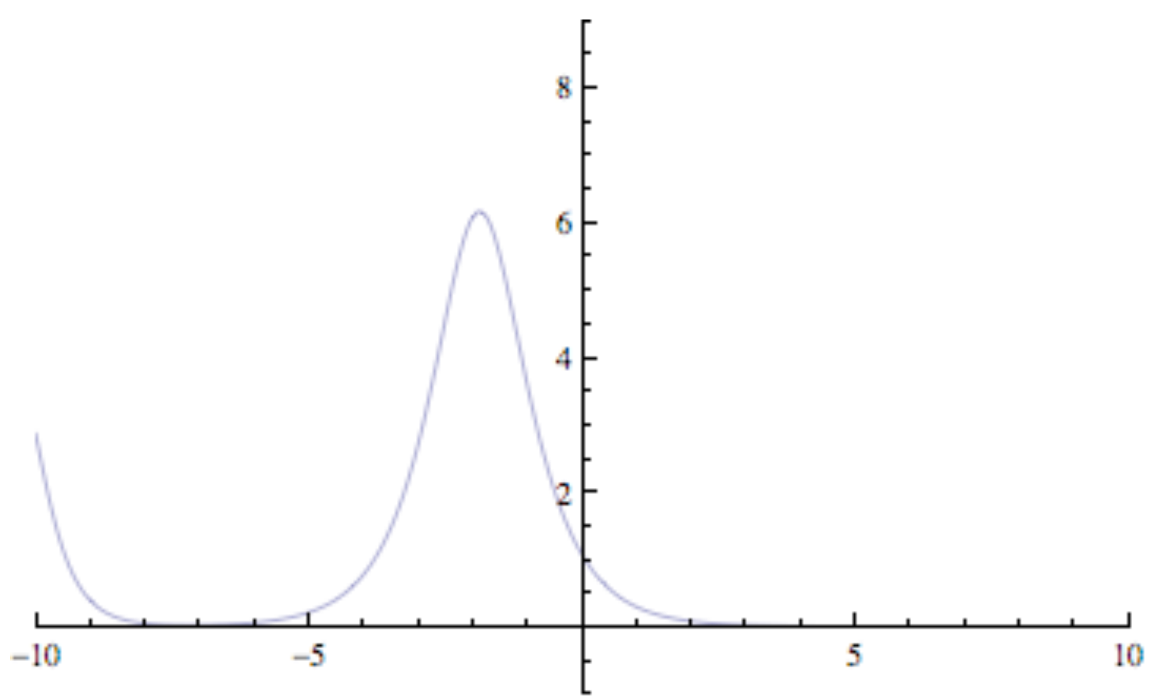
$$f_j^{(i)} = \alpha_i p_i^j e^{\eta_i} + \beta_i (-p_i)^j e^{\xi_i}$$

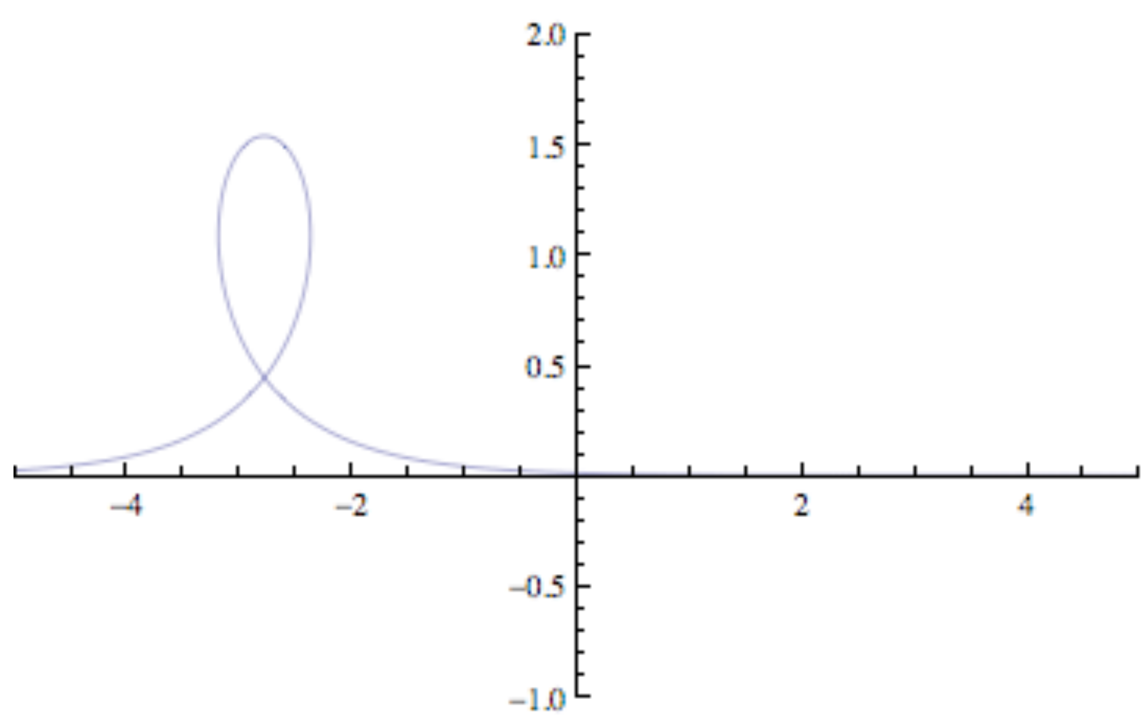
$$\eta_i = p_i s - 4p_i^3 t + \frac{1}{p_i} y$$

$$\xi_i = -p_i s + 4p_i^3 t - \frac{1}{p_i} y$$

$$\alpha_k, p_k \in \mathbb{R}, \quad \beta_k \in \sqrt{-1}\mathbb{R},$$

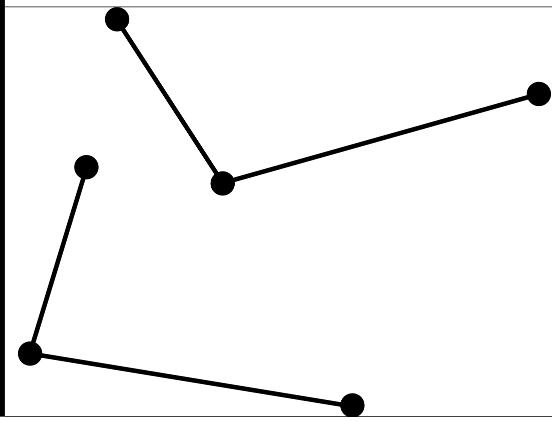
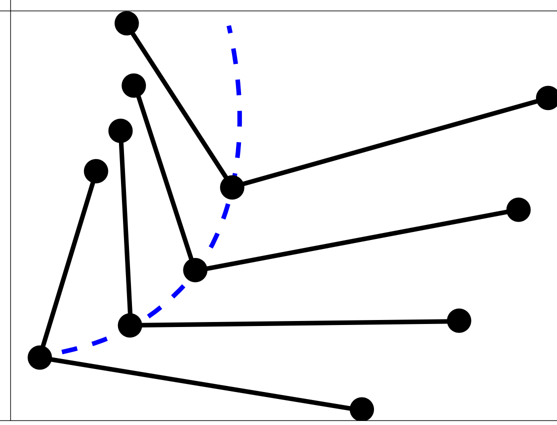
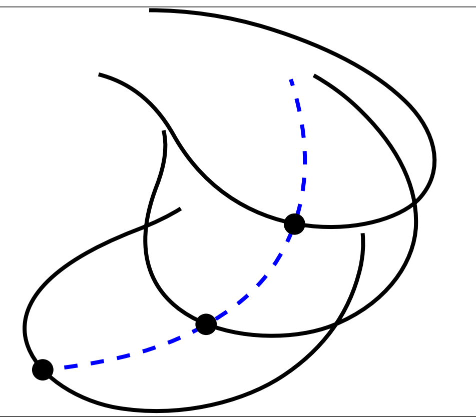
$$\theta = \frac{2}{\sqrt{-1}} \log \frac{\tau^*}{\tau}, \quad \gamma = \begin{bmatrix} x - \frac{1}{2}(\log \tau\tau^*)_y \\ \frac{1}{2\sqrt{-1}} \left(\log \frac{\tau^*}{\tau}\right)_y \end{bmatrix}$$

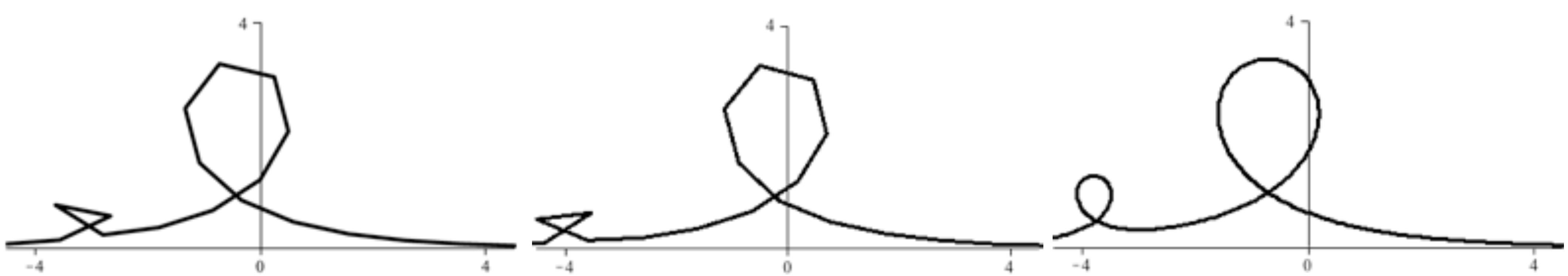




Purpose of this lecture

- **Formulation of motions of plane discrete curves preserving integrable structure**

equation	discrete	semi-discrete	continuous
curve	discrete	discrete	smooth
motion	discrete	continuous	continuous
schematic picture			



Discrete potential modified KdV equations

discrete potential mKdV

$$\tan\left(\frac{\theta_{n+1}^{m+1}-\theta_n^m}{4}\right)=\frac{b_m+a_n}{b_m-a_n}\tan\left(\frac{\theta_n^{m+1}-\theta_{n+1}^m}{4}\right)$$

$$\zeta=(n+m)\,\delta,\quad l=n-m,$$

$$\delta=a+b,\quad \epsilon=a-b,\quad \delta\to 0$$

semi-discrete potential mKdV

$$\frac{d}{d\zeta}\theta_l=\frac{2}{\epsilon}\tan\left(\frac{\theta_{l+1}-\theta_{l-1}}{4}\right)$$

$$s=l\epsilon+\zeta,\quad t=-\frac{\epsilon^3}{6}\zeta,\quad \epsilon\to 0$$

potential mKdV

$$\theta_t+\frac{1}{2}(\theta_s)^3+\theta_{sss}=0$$

R. Hirota, J. Phys.Soc. Jpn. **35**(1973) 289-294 (semi-discrete)

R. Hirota, J. Phys. Soc.Jpn. **67**(1998) 2234-2236 (discrete)

Continuous Limits (I)

semi-discrete potential mKdV

$$\frac{d}{d\zeta}\theta_l = \frac{2}{\epsilon}\tan\left(\frac{\theta_{l+1}-\theta_{l-1}}{4}\right)$$

$$s = l\epsilon + \zeta, \quad t = -\frac{\epsilon^3}{6}\zeta, \quad \epsilon \to 0$$

$$\frac{\partial}{\partial\zeta} = \frac{\partial s}{\partial\zeta}\frac{\partial}{\partial s} + \frac{\partial t}{\partial\zeta}\frac{\partial}{\partial t} = \frac{\partial}{\partial s} - \frac{\epsilon^3}{6}\frac{\partial}{\partial t}$$

$$\frac{\theta_{l+1}-\theta_{l-1}}{4} = \frac{\theta(s+\epsilon,t)-\theta(s-\epsilon,t)}{4} = \frac{1}{4}\left[\left(\theta + \epsilon\theta_s + \frac{\epsilon^2}{2}\theta_{ss} + \frac{\epsilon^3}{6}\theta_{sss} + \cdots\right) - \left(\theta - \epsilon\theta_s + \frac{\epsilon^2}{2}\theta_{ss} - \frac{\epsilon^3}{6}\theta_{sss} + \cdots\right)\right]$$

$$= \frac{\epsilon}{2}\theta_s + \frac{\epsilon^3}{12}\theta_{sss} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots \longrightarrow \tan\left(\frac{\theta_{l+1}-\theta_{l-1}}{4}\right) = \frac{\epsilon}{2}\theta_s + \left(\frac{1}{12}\theta_{sss} + \frac{1}{24}(\theta_s)^3\right)\epsilon^3 + \cdots$$

$$\theta_s - \frac{\epsilon^3}{6}\theta_t = \frac{2}{\epsilon}\left[\frac{\epsilon}{2}\theta_s + \left(\frac{1}{12}\theta_{sss} + \frac{1}{24}(\theta_s)^3\right)\epsilon^3 + \cdots\right]$$

$$\downarrow$$

$$\theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0$$

Continuous Limits (2)

discrete potential mKdV
$$\tan\left(\frac{\theta_{n+1}^{m+1}-\theta_n^m}{4}\right)=\frac{b_m+a_n}{b_m-a_n}\tan\left(\frac{\theta_n^{m+1}-\theta_{n+1}^m}{4}\right)$$

$$\zeta=(n+m)\,\delta,\quad l=n-m,$$
$$\delta=a+b,\quad \epsilon=a-b,\quad \delta\to 0$$

$$\tan\left(\frac{\theta_l^{\zeta+2\delta}-\theta_l^{\zeta}}{4}\right)=-\frac{\delta}{\epsilon}\tan\left(\frac{\theta_{l-1}^{\zeta+\delta}-\theta_{l+1}^{\zeta+\delta}}{4}\right)\quad\to\quad\tan\left(\frac{\theta_l^{\zeta+\delta}-\theta_l^{\zeta-\delta}}{4}\right)=-\frac{\delta}{\epsilon}\tan\left(\frac{\theta_{l-1}^{\zeta}-\theta_{l+1}^{\zeta}}{4}\right)$$

$$\frac{\theta_l^{\zeta+\delta}-\theta_l^{\zeta-\delta}}{4}=\frac{1}{4}\left\{\left(\theta_l+\delta\frac{d\theta_l}{d\zeta}+\cdots\right)-\left(\theta_l-\delta\frac{d\theta_l}{d\zeta}+\cdots\right)\right\}=\frac{\delta}{2}\frac{d\theta_l}{d\zeta}+\cdots,\qquad \tan x=x+\frac{x^3}{3}+\cdots$$

$$\tan\left(\frac{\theta_l^{\zeta+\delta}-\theta_l^{\zeta-\delta}}{4}\right)=-\frac{\delta}{\epsilon}\tan\left(\frac{\theta_{l-1}^{\zeta}-\theta_{l+1}^{\zeta}}{4}\right)\quad\to\quad\frac{\delta}{2}\frac{d\theta_l}{d\zeta}=\frac{\delta}{\epsilon}\tan\left(\frac{\theta_{l+1}-\theta_{l-1}}{4}\right)$$

$$\downarrow$$

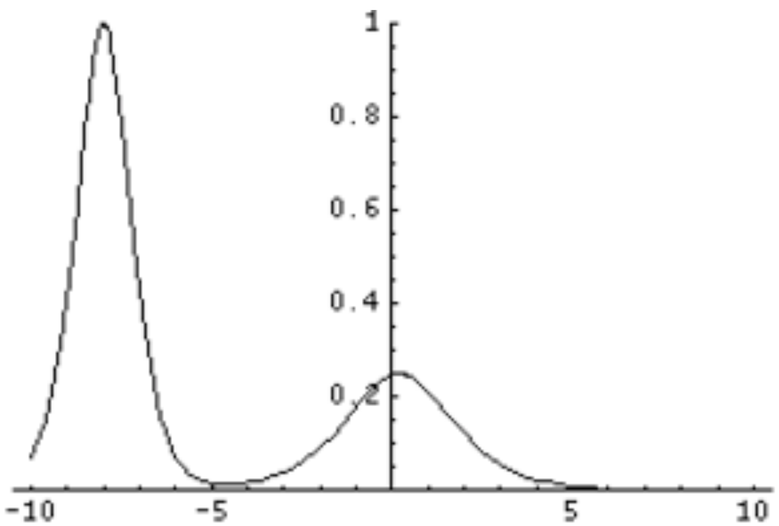
semi-discrete potential mKdV
$$\frac{d}{d\zeta}\theta_l=\frac{2}{\epsilon}\tan\left(\frac{\theta_{l+1}-\theta_{l-1}}{4}\right)$$

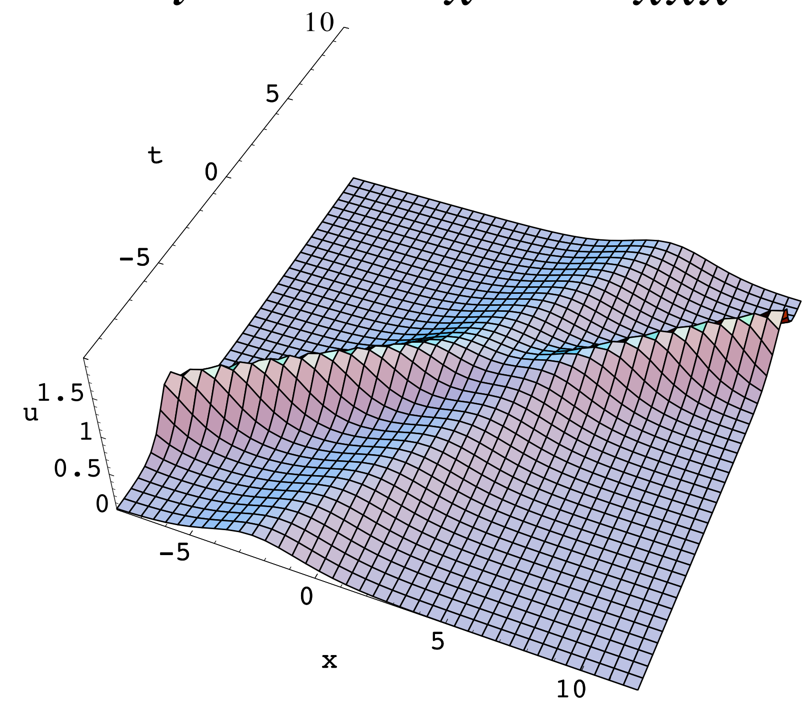
Rouch Sketch of Solitons and Integrable Systems

Solitons and integrability (I)

Solitons: solitary waves with character of particle

Korteweg-de Vries (KdV) equation $$u_t + 6uu_x + u_{xxx} = 0$$





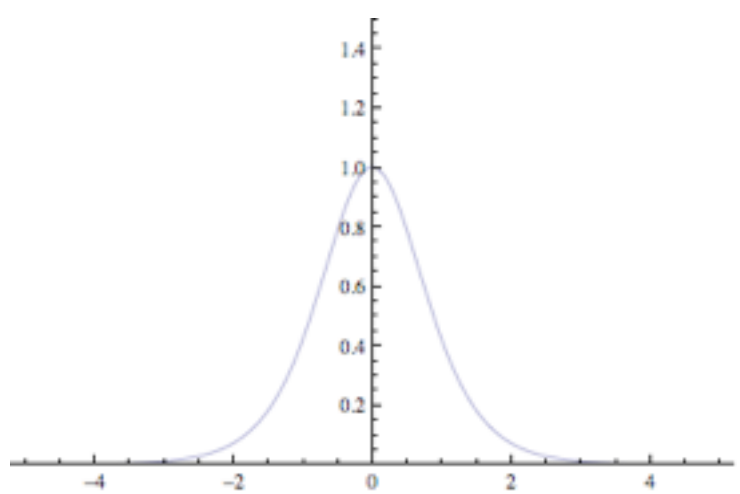
Physics: miraculous balance of nonlinearity and dispersion

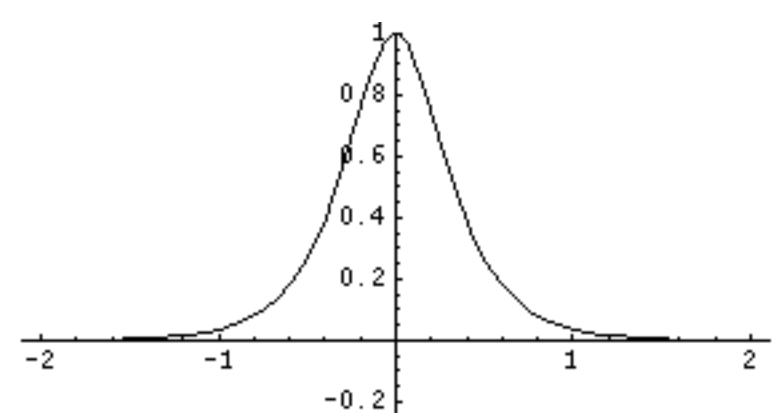
Nonlinearity: $u_t + 6uu_x = 0$

Dispersion: $u_t + u_{xxx} = 0$

Formal solution: $u(x,t) = f(x - 6ut)$

$$u(x,t) = \int_{-\infty}^{\infty} e^{ikx+i\omega t}\rho(k)\,dk, \quad \omega = k^3$$





Solitons and integrability (2)

- **Mathematics:** miraculous mathematical structure "**integrability**"

Typical features:

- Sufficiently (infinitely) many conserved quantities and (generalized Lie) symmetries

- Exact solvability by various methods

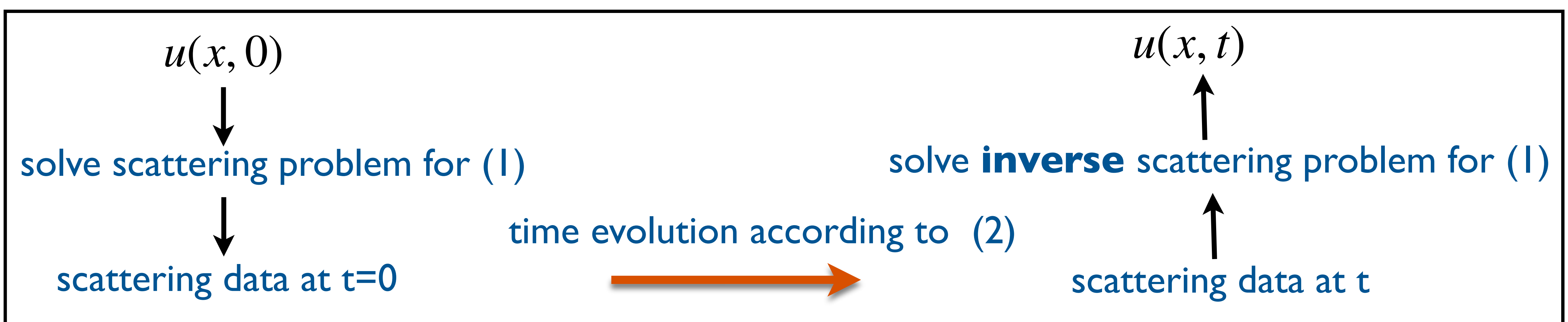
Inverse scattering method: $u_t + 6uu_x + u_{xxx} = 0$

auxiliary linear problem:

$$(1) \quad L\psi = \lambda\psi, \quad L = -\partial_x^2 + u$$

$$(2) \quad \psi_t = B\psi, \quad B = -\partial_x^3 + 3(u+\lambda)\partial_x$$

compatibility condition: $L_t = [B, L] \;\rightarrow\; u_t + 6uu_x + u_{xxx} = 0$



Solitons and integrability (3)

Wide class of exact solutions with good structure:

Soliton solutions, rational solutions (**determinant or pfaffian**),

quasi-periodic solutions (**theta functions**)

Kadomtsev-Petviashvili (KP) equation

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0$$

$$u = 2(\log \tau)_{xx}$$

$$\tau = \begin{vmatrix} f^{(1)} & \partial_x f^{(1)} & \cdots & \partial_x^{N-1} f^{(1)} \\ f^{(2)} & \partial_x f^{(2)} & \cdots & \partial_x^{N-1} f^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f^{(1)} & \partial_x f^{(1)} & \cdots & \partial_x^{N-1} f^{(1)} \end{vmatrix}$$

$$\partial_y f^{(k)} = \partial_x^2 f^{(k)}, \quad \partial_t f^{(k)} = \partial_x^3 f^{(k)}$$

e.g.,

$$f^{(k)} = \alpha_k e^{\eta_k} + \beta_k e^{\xi_k}$$

$$\eta_k = p_k x + p_k^2 y + p_k^3 t, \quad \xi_k = q_k x + q_k^2 y + q_k^3 t,$$

↓

N-soliton solution

Solitons and integrability (4)

Origin of integrability: Sato Theory (1981)

infinite dimensional space with infinite dimensional symmetry

KP equation $$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0$$

$$u = 2(\log\tau)_{xx}$$

$$\tau = \begin{vmatrix} f^{(1)} & f^{(2)} & \cdots & f^{(N)} \\ \partial_x f^{(1)} & \partial_x f^{(2)} & \cdots & \partial_x f^{(N)} \\ \vdots & \vdots & \cdots & \vdots \\ \partial_x^{N-1} f^{(1)} & \partial_x^{N-1} f^{(2)} & \cdots & \partial_x^{N-1} f^{(N)} \end{vmatrix}$$

$$f^{(i)} = \sum_{k=1}^{M} a_{ik} e^{\theta_k}, \quad \theta_k = p_k x + p_k^2 t + p_k^3 t$$

$$\downarrow$$

$$\tau = \det(A\Theta P)$$

$$A = (a_{ij})_{j=1,\dots,M}^{i=1,\dots,N}$$ N x M coefficient matrix

$$\partial_y f^{(i)} = \partial_x^2 f^{(i)}, \quad \partial_t f^{(i)} = \partial_x^3 f^{(i)}$$

$$\Theta = \mathrm{diag}(e^{\theta_1}, \cdots, e^{\theta_M}), \quad P = (p_i^{j-1})$$

$$G \in GL(N), \quad A \to A' = GA, \quad \tau \to \det G \times \tau \longrightarrow A \in GM(N, M)$$

$$\downarrow$$

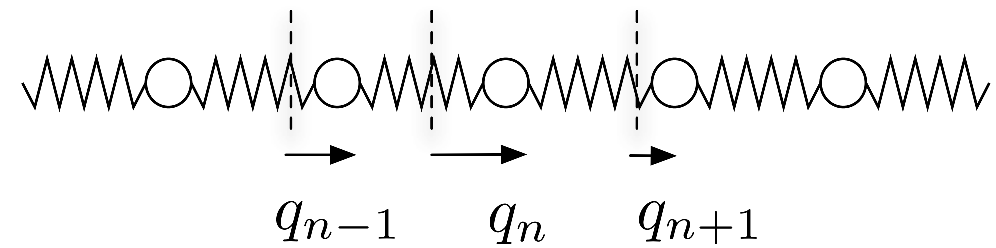
Solution space of KP: Universal Grassmannian manifold with GL(∞) symmetry

Brief Introduction to the Theory of Integrable Systems through “Toda Lattice”

Toda Lattice

Toda Lattice Equation: $$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$$

Relative displacement: $r_n = q_n - q_{n-1}$



potential energy: $\phi(r) \implies$ force: $-\phi'(r)$

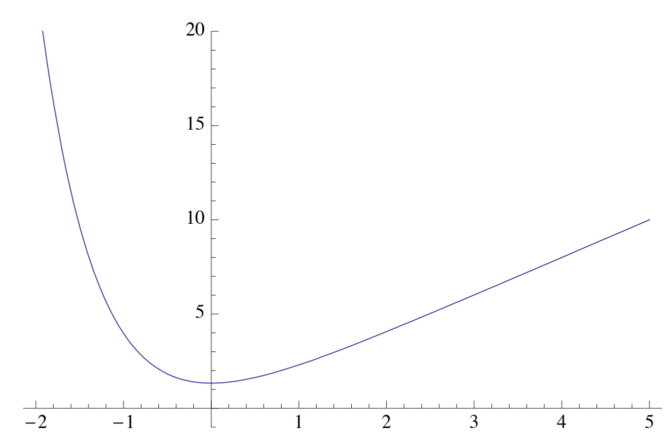
equation of motion: $$m\frac{d^2 q_n}{dt^2} = -\phi'(r_n) + \phi'(r_{n+1})$$

- **Hooke's Law:** $\phi(r) = \frac{1}{2}\kappa r^2$ force: $-\phi'(r) = -kr$

equation of motion: $$\frac{d^2 q_n}{dt^2} = -\kappa(q_n - q_{n-1}) + \kappa(q_{n+1} - q_n) = \kappa(q_{n+1} + q_{n-1} - 2q_n)$$

- **Toda potential:** $\phi(r) = \frac{a}{b}e^{-br} + ar \quad a, b > 0$ force: $-\phi'(r) = a(e^{-br} - 1)$

Remark: $r \sim 0 : \quad \phi(r) \sim \frac{a}{b} + \frac{ab}{2}r^2$ ~Hooke's Law



equation of motion: $$m\frac{d^2 q_n}{dt^2} = a\left[e^{-b(q_n - q_{n-1})} - e^{-b(q_{n+1}-q_n)}\right]$$

Variations of Toda Lattice

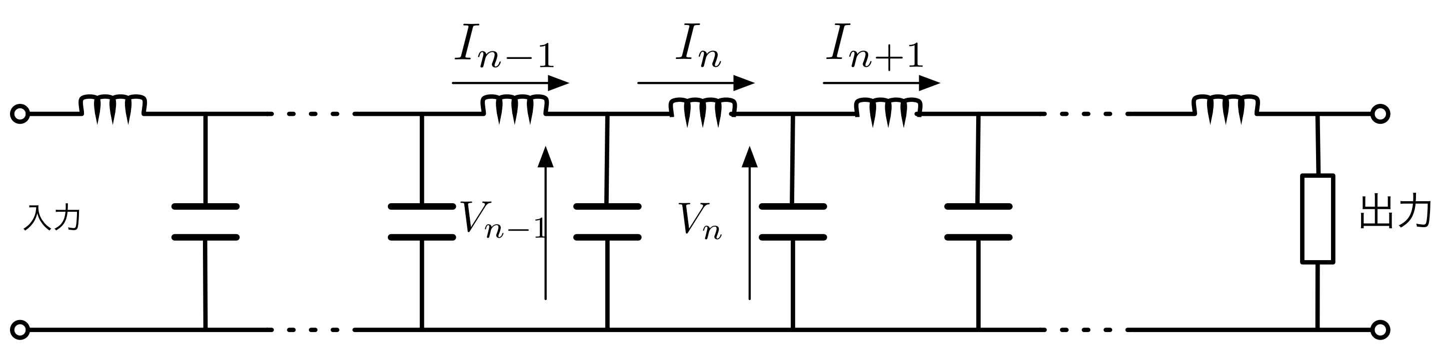
Toda Lattice Equation: $$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$$

$$\frac{d^2 r_n}{dt^2} = e^{r_{n+1}} + e^{r_{n-1}} - 2e^{r_n} \qquad r_n := q_n - q_{n+1}$$

$$\frac{d^2}{dt^2}\log(1+V_n) = V_{n+1} + V_{n-1} - 2V_n$$

or

$$\begin{cases} \dfrac{d}{dt}\log(1+V_n) = I_n - I_{n+1}, \\ \dfrac{dI_n}{dt} = V_{n-1} - V_n \end{cases} \qquad \begin{cases} 1 + V_n = e^{r_n}, \\ I_n = \dfrac{dq_n}{dt} \end{cases}$$



$$\begin{cases} \dfrac{da_n}{dt} = a_n(b_n - b_{n+1}), \\ \dfrac{db_n}{dt} = 2(a_{n-1}^2 - a_n^2) \end{cases} \qquad a_n = \frac{1}{2}e^{\frac{q_n - q_{n+1}}{2}}, \quad b_n = \frac{1}{2}\frac{dq_n}{dt}$$

Properties of Toda Lattice (I)

- Hamilton system of classical mechanics, with the Hamiltonian

$$H = \frac{1}{2m}\sum_n p_n^2 + \frac{a}{b}\sum_n e^{-b(q_n - q_{n-1})}, \quad q_n = q_n, \quad p_n = m\frac{dq_n}{dt},$$

- In the case of finite system with N particles (e.g. periodic system), there are N conserved quantities commuting w.r.t. the Poisson bracket. Namely, it is completely integrable systems, and the initial value problem can be solved by quadrature.

> **Liouville-Arnold's Theorem:**
> If a Hamilton system with N degrees of freedom possesses N conserved quantities commuting w.r.t. the Poisson bracket, then the initial value problem is solved by finite times applications of quadrature, namely,
> - arithmetic operations
> - differentiation & integration
> - taking inverse function
> - solving equations without differentiation

- Examples of completely integrable systems:
  - 2-body problem (Kepler problem)
  - Lagrange's top, Euler's top, Kowalevskaya top
  - Toda lattice (M. Toda, 1967)

Properties of Toda Lattice (2)

- Formulation as the spectral preserving deformation of an eigenvalue problem of a linear operator (Lax formalism):

$$\frac{dI_n}{dt} = V_{n-1} - V_n, \quad \frac{d}{dt}\log(1+V_n) = I_n - I_{n+1}, \quad n = 1,\ldots,N, \quad I_{N+1} = I_1, \quad V_{N+1} = V_1$$

$$L\Psi = \lambda\Psi, \quad L = \begin{pmatrix} I_1 & 1 & & & & 1+V_N \\ 1+V_1 & I_2 & 1 & & & \\ & 1+V_2 & I_3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1+V_{N-2} & I_{N-1} & 1 \\ 1 & & & & 1+V_{N-1} & I_N \end{pmatrix}$$

$$\frac{d\Psi}{dt} = B\Psi, \quad B = \begin{pmatrix} 0 & & & & & 1+V_N \\ 1+V_1 & 0 & & & & \\ & 1+V_2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1+V_{N-2} & 0 & \\ & & & & 1+V_{N-1} & 0 \end{pmatrix}$$

- Compatibility condition with λt = 0:

$$\frac{dL}{dt}\Psi + L\frac{d\Psi}{dt} = \lambda\frac{d\Psi}{dt} \;\rightarrow\; \frac{dL}{dt}\Psi + LB\Psi = BL\Psi \;\rightarrow\; \boxed{\frac{dL}{dt} = BL - LB} \;\Rightarrow$$ Toda Lattice

Properties of Toda Lattice (3)

$$(*)\begin{cases} \dfrac{dq_n}{dt} = \lambda e^{q_n-\bar{q}_n} + \dfrac{1}{\lambda}e^{\bar{q}_{n-1}-q_n} + \alpha \\ \dfrac{d\bar{q}_n}{dt} = \lambda e^{q_n-\bar{q}_n} + \dfrac{1}{\lambda}e^{\bar{q}_n-q_{n+1}} + \alpha \end{cases}$$

→ eliminate $\bar{q}_n(q_n)$ →

$$\begin{cases} \dfrac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \\ \dfrac{d^2\bar{q}_n}{dt^2} = e^{\bar{q}_{n-1}-\bar{q}_n} - e^{\bar{q}_n-\bar{q}_{n+1}} \end{cases}$$

Solving (*) for given q_n, we obtain another solution $\bar{q}_n$: Bäcklund transformation

- **Example:** $q_n = 0, \lambda = e^{-\kappa}, \alpha = -(e^{\kappa} + e^{-\kappa})$

$$\begin{cases} 0 = \lambda e^{-\bar{q}_n} + \dfrac{1}{\lambda}e^{\bar{q}_{n-1}} + \alpha \\ \dfrac{d\bar{q}_n}{dt} = \lambda e^{-\bar{q}_n} + \dfrac{1}{\lambda}e^{\bar{q}_n} + \alpha \end{cases} \rightarrow e^{\bar{q}_n} = X_n \rightarrow \begin{cases} X_n = -\dfrac{e^{-\kappa}}{e^{\kappa}X_{n-1} - (e^{\kappa}+e^{-\kappa})} & \text{discrete Riccati eq.} \\ X_n' = e^{\kappa}X_n^2 - (e^{\kappa}+e^{-\kappa})X_n + e^{-\kappa} & \text{Riccati eq.} \end{cases}$$

→

$$\boxed{X_n = \frac{1+e^{2\kappa(n-1)+2\beta t}}{1+e^{2\kappa n+2\beta t}}, \quad \beta = \sinh\kappa = \frac{e^{\kappa}-e^{-\kappa}}{2}}$$

- BT implies rich underlying mathematical structure
- BT can be formulated as the canonical transformation of the Hamilton system

Construction of solutions: Hirota method (I)

travelling wave solution
(1-soliton solution)

$$q_n = \frac{1+e^{2\kappa(n-1)+2\beta t}}{1+e^{2\kappa n+2\beta t}}, \quad \beta = \sinh\kappa = \frac{e^{\kappa}-e^{-\kappa}}{2}$$

$$e^{q_n} = \frac{\tau_{n-1}}{\tau_n} \quad \text{or} \quad q_n = \log\frac{\tau_{n-1}}{\tau_n}$$

$$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \;\rightarrow\; \frac{d^2}{dt^2}\log\tau_{n-1} - \frac{d^2}{dt^2}\log\tau_n = \frac{\tau_{n-2}\tau_n}{\tau_{n-1}^2} - \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}$$

$$\rightarrow \frac{d^2}{dt^2}\log\tau_{n-1} - \frac{\tau_{n-2}\tau_n}{\tau_{n-1}^2} = \frac{d^2}{dt^2}\log\tau_n - \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} = f(t) \;\rightarrow\; \tau_n''\tau_n - \tau_n'^2 = \tau_{n-1}\tau_{n+1} - f(t)\,\tau_n^2 \qquad (**)$$

Hirota's bilinear differential operator (D-operator)

$$D_x^m D_t^n f\cdot g = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n f(x,t)g(x',t')\Big|_{x=x',t=t'}$$

$$D_x f\cdot g = f_x g - f g_x, \quad D_x^2 f\cdot g = f_{xx}g - 2f_x g_x + f g_{xx}, \quad D_x D_t f\cdot g = f_{xt}g - f_x g_t - f_t g_x + f g_{xt}, \quad \text{etc.}$$

$$(**) \longrightarrow \frac{1}{2}D_t^2\,\tau_n\cdot\tau_n = \tau_{n+1}\tau_{n-1} - f(t)\,\tau_n^2$$

"Bilinear equation (form)" of Toda lattice

τ_n : **tau function**

Construction of solutions: Hirota method (2)

$$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \longrightarrow q_n = \log \frac{\tau_{n-1}}{\tau_n} \longrightarrow \frac{1}{2} D_t^2\, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - f(t)\, \tau_n^2$$

$$D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^m \, (\partial_t - \partial_{t'})^n \, f(x,t) g(x',t')\Big|_{x=x',t=t'}$$

Properties of D-operator:

Bilinearity:	$D_x^m D_t^n\, (af + bg) \cdot h = aD_x^m D_t^n\, f \cdot h + bD_x^m D_t^n\, g \cdot h$
Exchange rule:	$D_x^m D_t^n\, f \cdot g = (-1)^{m+n} D_x^m D_t^n\, g \cdot f$
constant argument:	$D_x^m D_t^n\, f \cdot 1 = \partial_x^m \partial_t^n\, f$
Rule for exponential fns.	$D_x^m D_t^n\, e^{p_1 x + q_1 t} \cdot e^{p_2 x + q_2 t} = (p_1 - p_2)^m (q_1 - q_2)^n\, e^{(p_1+p_2)x + (q_1+q_2)t}$

Construction of soliton solutions

- $q_n = 0$ is a solution. Correspondingly, $\tau_n = 1$ is a solution ($f(t)=1$).
- Apply perturabational technique to $\tau_n = 1$. Namely, assume the expansion

$$\tau_n = 1 + \epsilon f_n^{(1)} + \epsilon^2 f_n^{(2)} + \epsilon^3 f_n^{(3)} + \cdots$$

and plug it in the bilinear equation. Solve the equations obtained from coefficients of ε^j from the lower order. Stop this process at appropriate order and we have an approximate solution.

Construction of solutions: Hirota method (3)

$$\frac{1}{2}D_t^2\,\tau_n\cdot\tau_n=\tau_{n+1}\tau_{n-1}-\tau_n^2,\qquad \tau_n=1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}+\cdots$$

$$\frac{1}{2}D_t^2\,\left(1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}\right)\cdot\left(1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}\right)$$
$$=\left(1+\epsilon f_{n+1}^{(1)}+\epsilon^2 f_{n+1}^{(2)}+\epsilon^3 f_{n+1}^{(3)}\right)\left(1+\epsilon f_{n-1}^{(1)}+\epsilon^2 f_{n-1}^{(2)}+\epsilon^3 f_{n-1}^{(3)}\right)-\left(1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}\right)^2$$

$$O(\epsilon):\qquad f_n^{(1)\prime\prime}=f_{n+1}^{(1)}+f_{n-1}^{(1)}-2f_n^{(1)}$$

$$O(\epsilon^2):\quad f_n^{(2)\prime\prime}-f_{n+1}^{(2)}-f_{n-1}^{(2)}+2f_n^{(2)}=-\frac{1}{2}D_t^2\,f_n^{(1)}\cdot f_n^{(1)}+f_{n+1}^{(1)}f_{n-1}^{(1)}-f_n^{(1)2}$$

$$O(\epsilon^3):\qquad f_n^{(3)\prime\prime}-f_{n+1}^{(3)}-f_{n-1}^{(3)}+2f_n^{(3)}=-D_t^2\,f_n^{(1)}\cdot f_n^{(2)}+f_{n+1}^{(1)}f_{n-1}^{(2)}+f_{n+1}^{(2)}f_{n-1}^{(1)}-2f_n^{(1)}f_n^{(2)}$$

- **1-soliton solution:** $f_n^{(1)}=e^{\eta_1},\quad \eta_1=P_1 n+Q_1 t\ (+\text{const.})$

$$O(\epsilon):\quad Q_1^2=e^{P_1}+e^{-P_1}-2=(e^{\frac{P_1}{2}}-e^{-\frac{P_1}{2}})^2\ \rightarrow\ Q_1=\pm 2\sinh\frac{P_1}{2}$$

$$O(\epsilon^2):\quad f_n^{(2)\prime\prime}-f_{n+1}^{(2)}-f_{n-1}^{(2)}+2f_n^{(2)}=-\frac{1}{2}D_t^2\,f_n^{(1)}\cdot f_n^{(1)}+f_{n+1}^{(1)}f_{n-1}^{(1)}-f_n^{(1)2}=0$$

→ We can choose $f_n^{(2)}=0$. Similarly, we have $f_n^{(k)}=0\quad (k=3,4,\ldots.)$

→ Perturbation is truncated! We have an EXACT solution!

Construction of solutions: Hirota method (4)

1-soliton solution: $\tau_n = 1 + e^{\eta_1}, \quad \eta_1 = P_1 n \pm 2\sinh\frac{P_1}{2}t$

- **2-soliton solution:** $f_n^{(1)} = e^{\eta_1} + e^{\eta_2}, \quad \eta_i = P_i n + Q_i t \ (+\text{const.})$

$$O(\epsilon): \quad f_n^{(1)\prime\prime} = f_{n+1}^{(1)} + f_{n-1}^{(1)} - 2f_n^{(1)} \longrightarrow Q_i = \pm 2\sinh\frac{P_i}{2}$$

$$O(\epsilon^2): \quad f_n^{(2)\prime\prime} - f_{n+1}^{(2)} - f_{n-1}^{(2)} + 2f_n^{(2)} = -\frac{1}{2}D_t^2\, f_n^{(1)}\cdot f_n^{(1)} + f_{n+1}^{(1)}f_{n-1}^{(1)} - f_n^{(1)2}$$

$$= -\frac{1}{2}D_t^2\,(e^{\eta_1}+e^{\eta_2})\cdot(e^{\eta_1}+e^{\eta_2}) + \left(e^{\eta_1+P_1}+e^{\eta_2+P_2}\right)\left(e^{\eta_1-P_1}+e^{\eta_2-P_2}\right) - (e^{\eta_1}+e^{\eta_2})^2$$

$$= -D_t^2\, e^{\eta_1}\cdot e^{\eta_2} + e^{\eta_1+\eta_2+p_1-p_2} + e^{\eta_1+\eta_2-p_1+p_2} - 2e^{\eta_1+\eta_2}$$

$$= -(Q_1-Q_2)^2 e^{\eta_1+\eta_2} + \left(e^{\frac{P_1-P_2}{2}} - e^{-\frac{P_1-P_2}{2}}\right)^2 e^{\eta_1+\eta_2} = -\left(e^{\frac{P_1}{2}} - e^{-\frac{P_1}{2}}\right)\left(e^{\frac{P_2}{2}} - e^{-\frac{P_2}{2}}\right)\left(e^{\frac{P_1-P_2}{4}} - e^{-\frac{P_1-P_2}{4}}\right)^2 e^{\eta_1+\eta_2}$$

Put $f_n^{(2)} = A_{12}e^{\eta_1+\eta_2}$

$$\text{LHS} = A_{12}(Q_1+Q_2)^2 e^{\eta_1+\eta_2} - A_{12}\left(e^{\frac{P_1+P_2}{2}} - e^{-\frac{P_1+P_2}{2}}\right)^2 e^{\eta_1+\eta_2}$$

$$= -A_{12}\left(e^{\frac{P_1}{2}} - e^{-\frac{P_1}{2}}\right)\left(e^{\frac{P_2}{2}} - e^{-\frac{P_2}{2}}\right)\left(e^{\frac{P_1+P_2}{4}} - e^{-\frac{P_1+P_2}{4}}\right)^2 e^{\eta_1+\eta_2}$$

$$\longrightarrow \quad A_{12} = \left(\frac{e^{\frac{P_1-P_2}{4}} - e^{-\frac{P_1-P_2}{4}}}{e^{\frac{P_1+P_2}{4}} - e^{-\frac{P_1+P_2}{4}}}\right)^2 = \left(\frac{\sinh\frac{P_1-P_2}{4}}{\sinh\frac{P_1+P_2}{4}}\right)^2$$

Construction of solutions: Hirota method (5)

$$f_n^{(1)} = e^{\eta_1} + e^{\eta_2}, \quad f_n^{(2)} = A_{12}e^{\eta_1+\eta_2}, \quad \eta_i = P_i n + Q_i t, \quad Q_i = \pm 2\sinh\frac{P_i}{2}$$

$$O(\epsilon^3): \quad f_n^{(3)\prime\prime} - f_{n+1}^{(3)} - f_{n-1}^{(3)} + 2f_n^{(3)} = -D_t^2\, f_n^{(1)}\cdot f_n^{(2)} + f_{n+1}^{(1)}f_{n-1}^{(2)} + f_{n+1}^{(2)}f_{n-1}^{(1)} - 2f_n^{(1)}f_n^{(2)}$$

$$= -D_t^2\,(e^{\eta_1}+e^{\eta_2})\cdot A_{12}e^{\eta_1+\eta_2} + \left(e^{\eta_1+P_1}+e^{\eta_2+P_2}\right)A_{12}e^{\eta_1+\eta_2-P_1-P_2} + \left(e^{\eta_1-P_1}+e^{\eta_2-P_2}\right)A_{12}e^{\eta_1+\eta_2+P_1+P_2}$$

$$-\,2\,(e^{\eta_1}+e^{\eta_2})\,A_{12}e^{\eta_1+\eta_2}$$

1st term of RHS:

$$D_t^2\,(e^{\eta_1}+e^{\eta_2})\cdot e^{\eta_1+\eta_2} = D_t^2\,e^{\eta_1}\cdot e^{\eta_1+\eta_2} + D_t^2\,e^{\eta_2}\cdot e^{\eta_1+\eta_2} = [Q_1-(Q_1+Q_2)]^2\,e^{2\eta_1+\eta_2} + [Q_2-(Q_1+Q_2)]^2\,e^{\eta_1+2\eta_2}$$

$$= Q_2^2\,e^{2\eta_1+\eta_2} + Q_1^2\,e^{\eta_1+2\eta_2}$$

RHS: $$A_{12}\left[-Q_2^2\,e^{2\eta_1+\eta_2} - Q_1^2\,e^{\eta_1+2\eta_2} + \left(e^{\frac{P_2}{2}} - e^{-\frac{P_2}{2}}\right)^2 e^{2\eta_1+\eta_2} + \left(e^{\frac{P_1}{2}} - e^{-\frac{P_1}{2}}\right)^2 e^{\eta_1+2\eta_2}\right] = 0$$

→ We can choose $f_n^{(3)} = 0$. Similarly, we have $f_n^{(k)} = 0 \quad (k = 4, 5, \ldots.)$

→ Perturbation is truncated again! We have EXACT 2-soliton solution!

2-soliton solution: $$\tau_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+e^{\eta_2}}, \quad A_{12} = \left(\frac{\sinh\frac{P_1-P_2}{4}}{\sinh\frac{P_1+P_2}{4}}\right)^2$$

Determinant Structure of τ Function (I)

2-soliton solution:

$$\tau_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2},$$

$$\eta_i = P_i n \pm (e^{\frac{P_i}{2}} - e^{-\frac{P_i}{2}})t + \eta_{i0}, \quad A_{12} = \left(\frac{\sinh \frac{P_1-P_2}{4}}{\sinh \frac{P_1+P_2}{4}}\right)^2$$

Let $p_i := e^{\frac{P_i}{2}}$ $\longrightarrow$ $e^{\eta_i} = p_i^{2n}\, e^{(p_i - \frac{1}{p_i})t + \eta_{i0}}, \quad A_{12} = \left(\frac{p_1 - p_2}{p_1 p_2 - 1}\right)^2$

Determinant formula for 2-soliton solution:

$$\tau_n \doteqdot \begin{vmatrix} \varphi_n^{(1)} & \varphi_{n+1}^{(1)} \\ \varphi_n^{(2)} & \varphi_{n+1}^{(2)} \end{vmatrix}, \quad \varphi_n^{(i)} = e^{\xi_i} + e^{-\xi_i}, \quad e^{\xi_i} = p_i^n\, e^{\frac{1}{2}(p_i - \frac{1}{p_i})t + \xi_{i0}} = e^{\frac{\eta_i}{2}}$$

$$\begin{vmatrix} e^{\xi_1} + e^{-\xi_1} & p_1\, e^{\xi_1} + \frac{1}{p_1}\, e^{-\xi_1} \\ e^{\xi_2} + e^{-\xi_2} & p_2\, e^{\xi_2} + \frac{1}{p_2}\, e^{-\xi_2} \end{vmatrix} = (p_2 - p_1)\, e^{\xi_1+\xi_2} + (\frac{1}{p_2} - p_1)\, e^{\xi_1-\xi_2} + (p_2 - \frac{1}{p_1})\, e^{-\xi_1+\xi_2} + (\frac{1}{p_2} - \frac{1}{p_1})\, e^{-\xi_1-\xi_2}$$

$$= (\frac{1}{p_2} - \frac{1}{p_1})\, e^{-\xi_1-\xi_2}\left[1 + \frac{(1-p_1p_2)p_1}{(p_1-p_2)}\, e^{2\xi_1} + \frac{(p_1p_2-1)p_2}{(p_1-p_2)}\, e^{2\xi_2} - p_1p_2\, e^{2\xi_1+2\xi_2}\right]$$

$$\frac{(1-p_1p_2)p_1}{(p_1-p_2)}\, e^{2\xi_1} \to e^{2\xi_1}, \qquad \frac{(p_1p_2-1)p_2}{(p_1-p_2)}\, e^{2\xi_2} \to e^{2\xi_2}$$

$$\doteqdot 1 + e^{2\xi_1} + e^{2\xi_2} + \left(\frac{p_1 - p_2}{p_1p_2 - 1}\right)^2 e^{2\xi_1+2\xi_2}$$

Determinant Structure of τ Function (2)

Determinant formula for N-soliton solution:

τ function:

$$\tau_n = \begin{vmatrix} \varphi_n^{(1)} & \varphi_{n+1}^{(1)} & \cdots & \varphi_{n+N}^{(1)} \\ \varphi_n^{(2)} & \varphi_{n+1}^{(2)} & \cdots & \varphi_{n+N}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_n^{(N)} & \varphi_{n+1}^{(N)} & \cdots & \varphi_{n+N}^{(N)} \end{vmatrix}, \qquad \begin{aligned} \varphi_n^{(i)} &= e^{\xi_i} + e^{-\xi_i}, \\ e^{\xi_i} &= p_i^n \, e^{\frac{1}{2}(p_i - \frac{1}{p_i})t + \xi_{i0}} \end{aligned}$$

Bilinear equation:

$$\frac{1}{2} D_t^2 \, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

Dependent variable transformation:

$$q_n = \log \frac{\tau_{n-1}}{\tau_n}$$

Toda lattice equation:

$$\frac{d^2 q_n}{dt^2} = e^{q_{n-1} - q_n} - e^{q_n - q_{n+1}}$$

Two-dimensional Toda Lattice (I)

Two-dimensional Toda Lattice Equation: $$\frac{\partial^2 q_n}{\partial x \partial y} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$$

$$\frac{\partial^2 r_n}{\partial x \partial y} = e^{r_{n+1}} + e^{r_{n-1}} - 2e^{r_n}$$

$$\frac{\partial^2}{\partial x \partial y} \log(1+V_n) = V_{n+1} + V_{n-1} - 2V_n$$

or

$$\begin{cases} \dfrac{\partial}{\partial x} \log(1+V_n) = I_n - I_{n+1}, \\ \dfrac{\partial I_n}{\partial y} = V_{n-1} - V_n \end{cases}$$

Bilinear equation: $$\frac{1}{2} D_x D_y \, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

Dependent variable transformation:

$$q_n = \log \frac{\tau_{n-1}}{\tau_n}, \quad r_n = q_n - q_{n+1} = \log \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},$$

$$1 + V_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad I_n = \frac{\partial q_n}{\partial x} = \frac{\partial}{\partial x} \log \frac{\tau_{n-1}}{\tau_n}$$

- Relation to Toda lattice: t = x+y, s = x-y and impose $\dfrac{\partial q_n}{\partial s} = 0$

$$\frac{\partial^2 q_n}{\partial x \partial y} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \rightarrow \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) q_n = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \rightarrow \frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$$

Two-dimensional Toda Lattice (2)

Theorem: The following Casorati determinant

$$\tau_n = \begin{vmatrix} \varphi_n^{(1)} & \varphi_{n+1}^{(1)} & \cdots & \varphi_{n+N-1}^{(1)} \\ \varphi_n^{(2)} & \varphi_{n+1}^{(2)} & \cdots & \varphi_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_n^{(N)} & \varphi_{n+1}^{(N)} & \cdots & \varphi_{n+N-1}^{(N)} \end{vmatrix}$$

$$\frac{\partial \varphi_n^{(i)}}{\partial x} = \varphi_{n+1}^{(i)}, \qquad \frac{\partial \varphi_n^{(i)}}{\partial y} = -\varphi_{n-1}^{(i)}$$

satisfies the bilinear equation

$$\frac{1}{2} D_x D_y \, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

Remark:

$$\varphi_n^{(i)} = p_i^n \exp\left(p_i x - \frac{y}{p_k} + \eta_{i0}\right) + q_i^n \exp\left(q_i x - \frac{y}{q_i} + \xi_{i0}\right)$$ ⟶ N-soliton solution

Bilinear equation as Plücker relation (I)

- **Step 1:** derivative of τ = determinant with shifted columns

Freeman-Nimmo's notation:

$$\tau_n = \begin{vmatrix} \varphi_n^{(1)} & \varphi_{n+1}^{(1)} & \cdots & \varphi_{n+N-1}^{(1)} \\ \varphi_n^{(2)} & \varphi_{n+1}^{(2)} & \cdots & \varphi_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_n^{(N)} & \varphi_{n+1}^{(N)} & \cdots & \varphi_{n+N-1}^{(N)} \end{vmatrix} = |\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|, \quad \boldsymbol{j} = \begin{pmatrix} \varphi_{n+j}^{(1)} \\ \varphi_{n+j}^{(2)} \\ \vdots \\ \varphi_{n+j}^{(N)} \end{pmatrix}$$

Proposition (differential formula)

$$\tau_n = |\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,| \qquad \partial_x\tau_n = |\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}\,|$$

$$\tau_{n+1} = |\,\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1},\boldsymbol{N}\,| \qquad -\partial_y\tau_n = |\,\mathbf{-1},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|$$

$$\tau_{n-1} = |\,\mathbf{-1},\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2}\,| \qquad -\left(\partial_x\partial_y+1\right)\tau_n = |\,\mathbf{-1},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}\,|$$

Bilinear equation as Plücker relation (2)

- **Verification:** Left formulas are trivial. Noticing $\dfrac{\partial \varphi_n^{(i)}}{\partial x} = \varphi_{n+1}^{(i)}$

$$\partial_x \tau_n = \left| \mathbf{0}', \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \right| + \cdots + \left| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}', \mathbf{N-1} \right| + \left| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}' \right|$$

$$= | \mathbf{1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} | + \cdots + | \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}, \mathbf{N-1} | + | \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N} |$$

$$= \boxed{| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N} |}$$

Similarly, noticing $\dfrac{\partial \varphi_n^{(i)}}{\partial y} = -\varphi_{n-1}^{(i)}$

$$\partial_y \tau_n = \left| \mathbf{0}', \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \right| + \left| \mathbf{0}, \mathbf{1}', \cdots, \mathbf{N-2}, \mathbf{N-1} \right| + \cdots + \left| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}' \right|$$

$$= -| \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} | - | \mathbf{0}, \mathbf{0}, \cdots, \mathbf{N-2}, \mathbf{N-1} | - \cdots - | \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-2} |$$

$$= \boxed{-| \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} |}$$

$$\partial_x \partial_y \tau_n = -\left| \mathbf{-1}', \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \right| - \left| \mathbf{-1}, \mathbf{1}', \cdots, \mathbf{N-2}, \mathbf{N-1} \right| + \cdots + \left| \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}' \right|$$

$$= -| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} | - | \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N} |$$

$$= \boxed{-\tau_n - | \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N} |}$$

Bilinear equation as Plücker relation (3)

- **Step 2:** Bilinear equation = identity of determinant (Plücker relation)

$$0 = \frac{1}{2}D_xD_y\tau_n\cdot\tau_n - \tau_{n+1}\tau_{n-1} + \tau_n^2 = \left(\partial_x\partial_y\tau_n\right)\tau_n - (\partial_x\tau_n)\left(\partial_y\tau_n\right) - \tau_{n+1}\tau_{n-1} + \tau_n^2$$

$$= \left(-|\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,| - |\,\mathbf{-1},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}\,|\right)\times|\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|$$
$$-|\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}\,|\times\left(-|\,\mathbf{-1},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|\right)$$
$$-|\,\mathbf{1},\mathbf{2},\cdots,\boldsymbol{N-1},\boldsymbol{N}\,|\times|\,\mathbf{-1},\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2}\,|$$
$$+|\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|\times|\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|$$

$$= -|\,\mathbf{-1},\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2}\,|\times|\,\mathbf{1},\mathbf{2},\cdots,\boldsymbol{N-1},\boldsymbol{N}\,|$$
$$+|\,\mathbf{-1},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|\times|\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}\,|$$
$$-|\,\mathbf{-1},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}\,|\times|\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|$$

Bilinear Equation:

$$0 = |\,\mathbf{-1},\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2}\,| \times |\,\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1},\boldsymbol{N}\,|$$
$$+\ |\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,| \times |\,\mathbf{-1},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}\,|$$
$$-\ |\,\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}\,| \times |\,\mathbf{-1},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}\,|$$

Bilinear equation as Plücker relation (4)

Proposition: Laplace expansion of determinant

$A=(a_{ij})_{1\le i,j\le N}$: N x N matrix

$|A|^{i_1 i_2\cdots i_l}_{j_1 j_2\cdots j_l}$: ℓ x ℓ minor determinant obtained by choosing $i_1,i_2...,i_\ell$-th rows and $j_1,j_2,...,j_\ell$-th columns from A

$\overline{|A|}^{i_1 i_2\cdots i_l}_{j_1 j_2\cdots j_l}$: (N-ℓ)x (N-ℓ) minor determinant obtained by removing $i_1,i_2...,i_\ell$-th rows and $j_1,j_2,...,j_\ell$-th columns from A

Fix ℓ integers $i_1,i_2...,i_\ell$ such that $1\le i_1<i_2<\cdots<i_l\le N$ Then we have:

$$|A| = (-1)^{i_1+\cdots+i_l}\sum_{1\le j_1<\cdots<j_l\le N}(-1)^{j_1+\cdots+j_l}\ |A|^{i_1 i_2\cdots i_l}_{j_1 j_2\cdots j_l}\times\overline{|A|}^{i_1 i_2\cdots i_l}_{j_1 j_2\cdots j_l}$$

Example: ℓ = 1, i_1=1. $|A|^{i_1}_{j_1}=a_{1j_1}$

$$|A| = \sum_{1\le j_1\le N}(-1)^{1+j_1}a_{1j_1}\times\overline{|A|}^{1}_{j_1} = \sum_{1\le j_1\le N}a_{1j_1}\times A_{1j_1},\quad A_{1j_1}:\ (1,j_1)\text{-cofactor}$$

→ Expansion w.r.t. 1st row

Bilinear equation as Plücker relation (5)

Consider the following identity of 2N x 2N determinant:

$$0 = \left| \begin{array}{c:cccc:ccc:cc} -1 & 0 & 1 & \cdots & N-2 & & \emptyset & & N-1 & N \\ \hdashline -1 & & \emptyset & & & 1 & \cdots & N-2 & N-1 & N \end{array} \right|$$

$$\qquad\qquad \underbrace{\qquad\qquad\qquad}_{N-1} \qquad \underbrace{\qquad\qquad}_{N-2}$$

- Verification: Subtract the lower block from the upper block, then

$$\mathrm{RHS} = \left| \begin{array}{c:c|ccc|ccc:cc} \emptyset & 0 & 1 & \cdots & N-2 & -(1) & \cdots & -(N-2) & \emptyset & \emptyset \\ \hdashline -1 & & & \emptyset & & 1 & \cdots & N-2 & N-1 & N \end{array} \right|$$

$$= \left| \begin{array}{c:cccc:ccc:cc} \emptyset & 0 & 1 & \cdots & N-2 & & \emptyset & & \emptyset & \emptyset \\ \hdashline -1 & & \emptyset & & & 1 & \cdots & N-2 & N-1 & N \end{array} \right|$$

$$\qquad\qquad \underbrace{\qquad\qquad\qquad}_{N-1} \qquad \underbrace{\qquad\qquad}_{N-2}$$

Now apply the Laplace expansion with ℓ=N, i_l=1,...,i_N=N. Since the upper block contains N+1 empty columns, all the terms in the expansion are 0.

Bilinear equation as Plücker relation (6)

$$0 = \begin{vmatrix} -1 & 0 \quad 1 \quad \cdots \quad N-2 & \emptyset & N-1 \quad N \\ -1 & \emptyset & 1 \quad \cdots \quad N-2 & N-1 \quad N \end{vmatrix}$$

Apply the Laplace expansion to RHS directly: Plücker relation (simplest case)

$$\begin{aligned} 0 = &\ |-1, 0, 1, \cdots, N-2| \times |1, \cdots, N-2, N-1, N| \\ + &\ |0, 1, \cdots, N-2, N-1| \times |-1, 1, \cdots, N-2, N| \\ - &\ |0, 1, \cdots, N-2, N| \times |-1, 1, \cdots, N-2, N-1| \end{aligned}$$

→ Bilinear equation

Bilinear equation of 2DTL = Plücker relation

$$\tau_n = \begin{vmatrix} \varphi_n^{(1)} & \varphi_{n+1}^{(1)} & \cdots & \varphi_{n+N-1}^{(1)} \\ \varphi_n^{(2)} & \varphi_{n+1}^{(2)} & \cdots & \varphi_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_n^{(N)} & \varphi_{n+1}^{(N)} & \cdots & \varphi_{n+N-1}^{(N)} \end{vmatrix} \longrightarrow \frac{1}{2} D_x D_y\, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

$$\frac{\partial \varphi_n^{(i)}}{\partial x} = \varphi_{n+1}^{(i)}, \qquad \frac{\partial \varphi_n^{(i)}}{\partial y} = -\varphi_{n-1}^{(i)}$$

Essential Structure of Integrable Systems

- **Infinite number of Plücker relations**
  - Distinguished column vectors -1, 0, N-1,N can be arbitrary
  - Number of distinguished column vectors is arbitrary (more than 4)

- **Diffrential/difference structure:**

  With appropriate differential/difference structure in τ, any determinant with arbitrary shift can be obtained by applying suitable differential operator to τ.

Example: introduce an infinite number of independent variables x_j,y_j (j=1,2,...) such that $\frac{\partial \varphi_n^{(i)}}{\partial x_j} = \varphi_{n+j}^{(i)}, \quad \frac{\partial \varphi_n^{(i)}}{\partial y_j} = -\varphi_{n-j}^{(i)}$

- **Infinite number of Plücker relations**

  = Infinite number of bilinear equations sharing common solutions

  (with the above differential/difference structure) "2DTL hierarchy"

  (with x_j or y_j only) "KP hierarchy"

Sato Theory:

- Solution space of soliton equations is the universal Grassmann manifold
- τ functions are the Plücker coordinates.

Reductions (I)

- **Reduction:** Procedure to yield a new equation by restricting the solution space (parameters of solutions)

- **2DTL → 1DTL:** Put t = x+y, s = x-y and impose $\dfrac{\partial q_n}{\partial s} = 0$

$$\frac{\partial^2 q_n}{\partial x \partial y} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \rightarrow \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) q_n = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \rightarrow \boxed{\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}}$$

- **2DTL → sinh-Gordon:** impose 2-periodicity $q_{n+2} = q_n$

$$\frac{\partial^2 q_n}{\partial x \partial y} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \rightarrow \begin{cases} \dfrac{\partial^2 q_0}{\partial x \partial y} = e^{q_1 - q_0} - e^{q_0 - q_1} \\ \dfrac{\partial^2 q_1}{\partial x \partial y} = e^{q_0 - q_1} - e^{q_1 - q_0} \end{cases} \rightarrow \frac{\partial^2 v}{\partial x \partial y} = 2\left(e^{-v} - e^{v}\right), \quad v := q_0 - q_1$$

$$\longrightarrow \quad \boxed{\frac{\partial^2 v}{\partial x \partial y} = -4 \sinh v}$$

sinh-Gordon equation

$$\boxed{\frac{\partial^2 \theta}{\partial x \partial y} = -4 \sin \theta}$$

sine-Gordon equation $v = \sqrt{-1}\theta \in \sqrt{-1}\mathbb{R}$

Reductions (2)

- **2DTL → 1DTL:** Impose restriction on τ function to realize $\dfrac{\partial q_n}{\partial s} = 0$

$$q_n = \log \frac{\tau_{n-1}}{\tau_n} \quad \Longrightarrow \quad \partial_s q_n = \frac{\partial_s \tau_{n-1}}{\tau_{n-1}} - \frac{\partial_s \tau_n}{\tau_n} = 0 \quad \Longrightarrow \quad \boxed{\partial_s \tau_n = \text{const.} \times \tau_n}$$

Soliton solution: $\tau_n = \det\left(\varphi_{n+j-1}^{(i)}\right)_{i,j=1,\ldots,N}$

$$\varphi_n^{(i)} = p_i^n e^{p_i x - \frac{y}{p_i}} + q_i^n e^{q_i x - \frac{y}{q_i}} = p_i^n \exp\left[\frac{1}{2}\left(p_i - \frac{1}{p_i}\right)t + \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)s\right] + q_i^n \exp\left[\frac{1}{2}\left(q_i - \frac{1}{q_i}\right)t + \frac{1}{2}\left(q_i + \frac{1}{q_i}\right)s\right]$$

$$\to \partial_s \varphi_n^{(i)} = \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)p_i^n \exp\left[\frac{1}{2}\left(p_i - \frac{1}{p_i}\right)t + \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)s\right] + \frac{1}{2}\left(q_i + \frac{1}{q_i}\right)q_i^n \exp\left[\frac{1}{2}\left(q_i - \frac{1}{q_i}\right)t + \frac{1}{2}\left(q_i + \frac{1}{q_i}\right)s\right] \propto \varphi_n^{(i)}$$

$$\to p_i + \frac{1}{p_i} = q_i + \frac{1}{q_i} \quad \longrightarrow \quad (p_i - q_i)\left(1 - \frac{1}{p_i q_i}\right) = 0 \quad \longrightarrow \quad \boxed{q_i = \frac{1}{p_i}}, \quad \partial_s \varphi_n^{(i)} = \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)\varphi_n^{(i)}$$

$$\Longrightarrow \partial_s \tau_n = C_N\, \tau_n, \quad C_N = \sum_{i=1}^{N} \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)$$

Reductions (3)

$$t = x + y, \quad s = x - y, \quad q_i = \frac{1}{p_i}, \quad \partial_s \tau_n = C_N\, \tau_n, \quad C_N = \sum_{i=1}^{N} \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)$$

2DTL $\left(\partial_x \partial_y \tau_n\right)\tau_n - \left(\partial_x \tau_n\right)\left(\partial_y \tau_n\right) = \tau_{n+1}\tau_{n-1} - \tau_n^2$

$$\text{LHS} = \left(\partial_t^2 - \partial_s^2\right)\tau_n \times \tau_n - \left(\partial_t - \partial_s\right)\tau_n \times \left(\partial_t + \partial_s\right)\tau_n = \left(\partial_t^2 - C_N^2\right)\tau_n \times \tau_n - \left(\partial_t - C_N\right)\tau_n \times \left(\partial_t + C_N\right)\tau_n$$

$$= \left(\partial_t^2 \tau_n\right)\tau_n - \left(\partial_t \tau_n\right)^2 \quad \Longrightarrow \quad \boxed{\left(\partial_t^2 \tau_n\right)\tau_n - \left(\partial_t \tau_n\right)^2 = \tau_{n+1}\tau_{n-1} - \tau_n^2}$$

1DTL!

Bilinear equation and τ function for 1DTL:

$$\frac{1}{2} D_t^2\, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

$$\tau_n = \begin{vmatrix} \varphi_n^{(1)} & \varphi_{n+1}^{(1)} & \cdots & \varphi_{n+N-1}^{(1)} \\ \varphi_n^{(2)} & \varphi_{n+1}^{(2)} & \cdots & \varphi_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_n^{(N)} & \varphi_{n+1}^{(N)} & \cdots & \varphi_{n+N-1}^{(N)} \end{vmatrix} \qquad \varphi_n^{(i)} = p_i^n\, e^{\frac{t}{2}\left(p_i - \frac{1}{p_i}\right) + \eta_{i0}} + p_i^{-n}\, e^{-\frac{t}{2}\left(p_i - \frac{1}{p_i}\right) + \xi_{i0}}$$

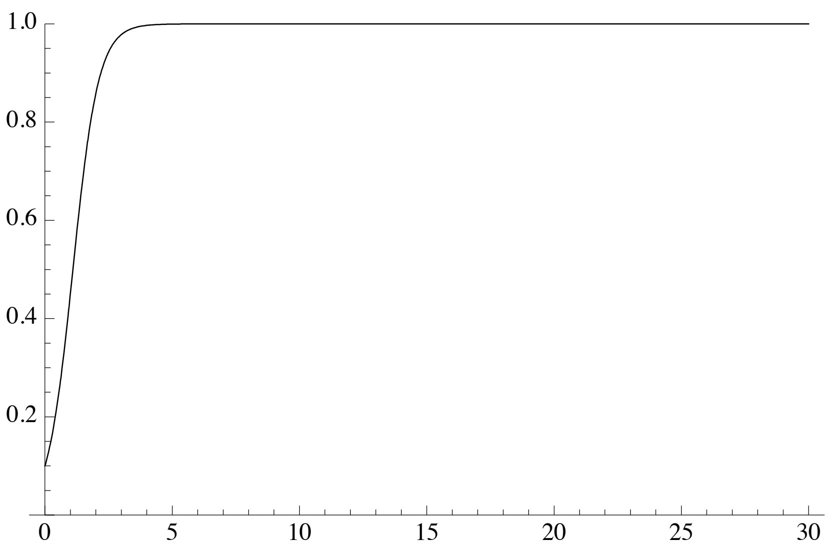
Discretization Preserving Integrability

Discretization preserving integrability (I)

Basic idea: the logistic equation

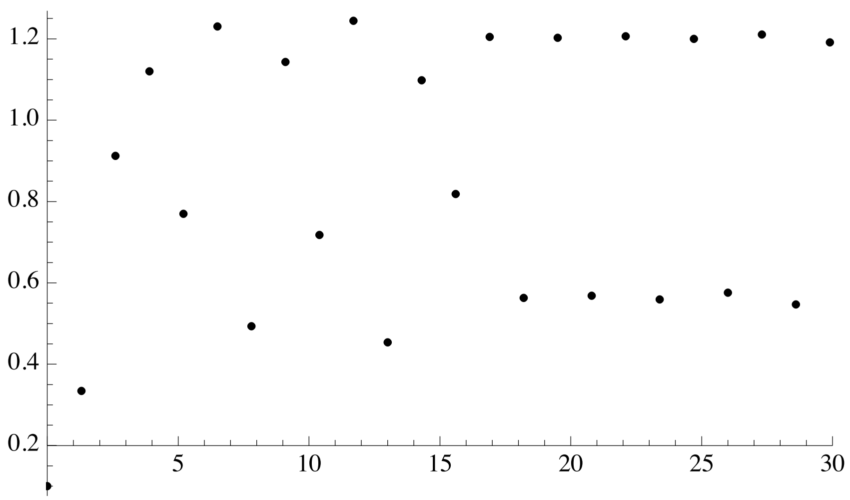
$$\frac{du}{dt} = au(1-u), \qquad a > 0$$

$$u = \frac{1}{1+Ce^{-at}}, \qquad C = \frac{1-u(0)}{u(0)}$$



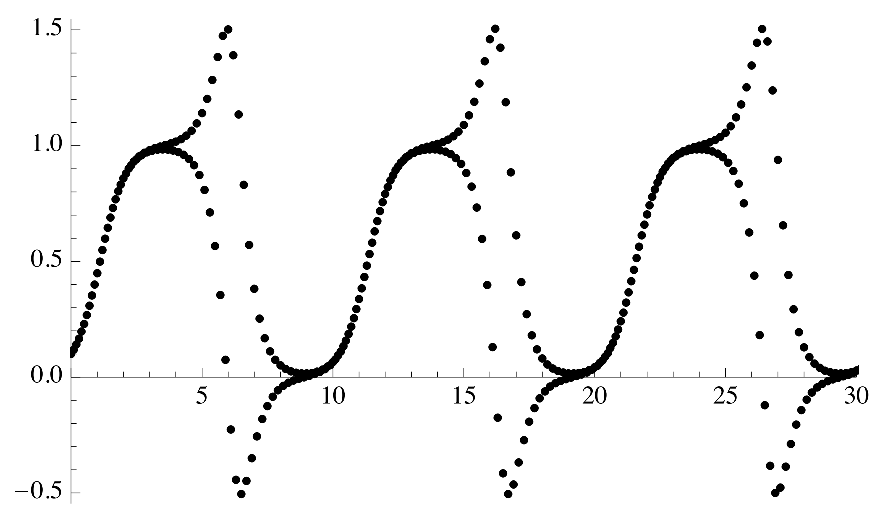
Three discretizations

$$\frac{u_{n+1}-u_n}{h} = au_n(1-u_n) \qquad \frac{u_{n+1}-u_{n-1}}{2h} = au_n(1-u_n) \qquad \frac{u_{n+1}-u_n}{h} = au_n(1-u_{n+1})$$

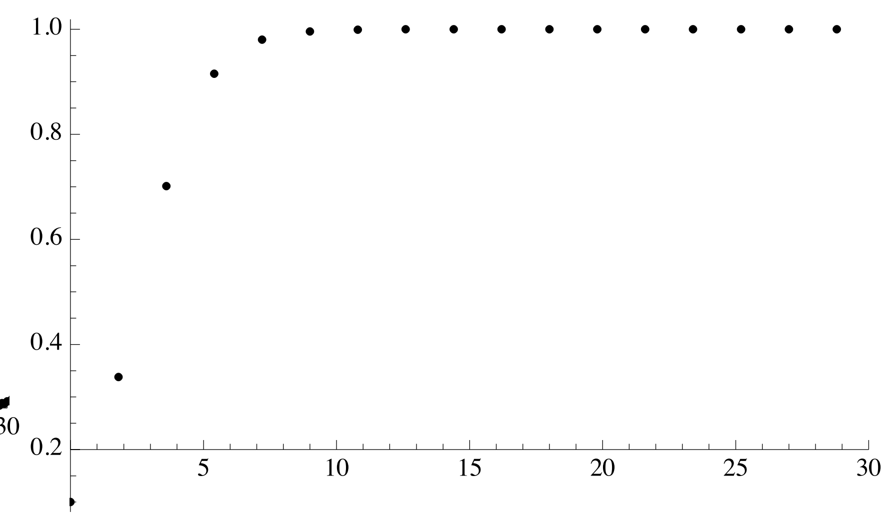


logistic map: chaotic

$$v_{n+1} = \alpha v_n(1-v_n)$$



chaotic



“integrable”

Discretization preserving integrability (2)

Logistic equation

$$\frac{du}{dt} = au(1-u)$$

↓

dependent variable transformation

$$u = \frac{1}{1+f}$$

$$f = \frac{1}{u} - 1$$

↓

linear eq. $$\frac{df}{dt} = -af$$

↓

solution $$f = Ce^{-at}$$

Discrete logistic equation

$$\frac{u_{n+1} - u_n}{h} = au_n(1 - u_{n+1})$$

↑

$$f_n = \frac{1}{u_n} - 1$$

↑

$$\frac{f_n - f_{n-1}}{h} = -af_n$$

$$f_n = (1+ah)^{-1} f_{n-1}$$

↓

$$f_n = C(1+ah)^{-n}$$

$$n = \frac{t}{h},\ t \to 0$$

Discretization preserving integrability (3)

Burgers equation

$$u_t = uu_x + \nu u_{xx}$$

$\downarrow$

Cole-Hopf transformation $\quad u = 2\nu\,(\log f)_x$

$\downarrow$

diffusion eq. $\quad f_t = \nu f_{xx}$

$\downarrow$

Shock wave sol. $\quad f = 1 + \sum_{k=1}^{N} e^{p_k x + \nu p_k^2 t + \eta_k}$

discrete Burgers equation

$$\frac{U_n^{t+1}}{U_n^t} = \frac{1 + \frac{1-2\alpha}{\alpha\gamma^2} U_n^t + \frac{1}{\gamma^2} U_n^t U_{n+1}^t}{1 + \frac{1-2\alpha}{\alpha\gamma^2} U_{n-1}^t + \frac{1}{\gamma^2} U_{n-1}^t U_n^t}$$

$\uparrow$

$$U_n^t = \gamma\,\frac{F_{n+1}^t}{F_n^t}$$

$\uparrow$

$$\frac{F_n^{t+1} - F_n^t}{\delta} = \nu\,\frac{F_{n+1}^t - 2F_n^t + F_{n-1}^t}{\epsilon^2}$$

$$F_n^{t+1} = \alpha(F_{n+1}^t + F_{n-1}^t) + (1-2\alpha)F_n^t, \quad \alpha = \frac{\nu\delta}{\epsilon^2}$$

$\downarrow$

$$F_n^t = 1 + \sum_{k=1}^{N} e^{q_k n + \omega_k t + \eta_k}$$

$$\omega_k = \log\left[1 + \alpha\left(e^{q_k} - 2 + e^{-q_k}\right)\right]$$

Discretization preserving integrability (4)

Two-dimensional Toda lattice

$$\frac{\partial^2 r_n}{\partial x \partial y} = e^{r_{n+1}} - 2e^{r_n} + e^{-r_{n-1}}$$

↓

dependent variable transformation

$$e^{r_n} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}$$

↓

Bilinear equation of Hirota type

$$\tau_{nxy}\tau_n - \tau_{nx}\tau_{ny} = \tau_{n+1}\tau_{n-1} - \lambda\tau_n^2$$

↓

τ function

$$\tau_n = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}$$

$$\boxed{\frac{\partial f_n^{(k)}}{\partial x} = f_{n+1}^{(k)}, \quad \frac{\partial f_n^{(k)}}{\partial y} = -f_{n+1}^{(k)}}$$

→

$$\boxed{\Delta_l f_n^{(k)}(l,m) = f_{n+1}^{(k)}(l,m), \quad \Delta_m f_n^{(k)}(l,m) = -f_{n-1}^{(k)}(l,m)}$$

$$\tau_n(l,m) = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}$$

↑

$$(1+ab)\tau_n(l+1,m+1)\tau_n(l,m) - \tau_n(l+1,m)\tau_n(l,m+1)$$
$$= ab\tau_{n+1}(l+1,m)\tau_{n-1}(l,m+1)$$

↑

$$e^{R_n(l,m)} = \frac{\tau_{n+1}(l+1,m)\tau_{n-1}(l,m+1)}{\tau_n(l+1,m)\tau_n(l,m+1)}$$

↑

$$\Delta_{+l}\Delta_{+m}R_n(l,m) = F_{n+1}(l+1,m) + F_{n-1}(l,m+1) - F_n(l+1,m) - F_n(l,m+1)$$

$$F_n(l,m) = \frac{1}{ab}\log\left[1 + abe^{R_n(l,m)}\right]$$

Discretization of 2DTL (I)

τ function of 2DTL $\tau_n = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}$

$$\frac{\partial f_n^{(k)}}{\partial x} = f_{n+1}^{(k)}, \quad \frac{\partial f_n^{(k)}}{\partial y} = -f_{n+1}^{(k)}$$

Bilinear equation $\tau_{nxy}\tau_n - \tau_{nx}\tau_{ny} = \tau_{n+1}\tau_{n-1} - \tau_n^2$

- **Discretize on the level of linear equation**
- **Preserve the determinant structure**

$$\tau_n(l,m) = \begin{vmatrix} f_n^{(1)}(l,m) & f_{n+1}^{(1)}(l,m) & \cdots & f_{n+N-1}^{(1)}(l,m) \\ f_n^{(2)}(l,m) & f_{n+1}^{(2)}(l,m) & \cdots & f_{n+N-1}^{(2)}(l,m) \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)}(l,m) & f_{n+1}^{(N)}(l,m) & \cdots & f_{n+N-1}^{(N)}(l,m) \end{vmatrix}$$

$$\frac{f_n^{(i)}(l,m) - f_n^{(i)}(l-1,m)}{a} = f_{n+1}^{(i)}(l,m)$$

$$\frac{f_n^{(i)}(l,m) - f_n^{(i)}(l,m-1)}{b} = -f_{n-1}^{(i)}(l,m)$$

↓ Plücker relation + difference formula

Bilinear equation

$$(1+ab)\tau_n(l+1,m+1)\tau_n(l,m) - \tau_n(l+1,m)\tau_n(l,m+1) = ab\tau_{n+1}(l+1,m)\tau_{n-1}(l,m+1)$$

Discretization of 2DTL (2)

$$\tau_n(l,m) = \begin{vmatrix} f_n^{(1)}(l,m) & f_{n+1}^{(1)}(l,m) & \cdots & f_{n+N-1}^{(1)}(l,m) \\ f_n^{(2)}(l,m) & f_{n+1}^{(2)}(l,m) & \cdots & f_{n+N-1}^{(2)}(l,m) \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)}(l,m) & f_{n+1}^{(N)}(l,m) & \cdots & f_{n+N-1}^{(N)}(l,m) \end{vmatrix}$$

$$\frac{f_n^{(i)}(l,m) - f_n^{(i)}(l-1,m)}{a} = f_{n+1}^{(i)}(l,m)$$

$$\frac{f_n^{(i)}(l,m) - f_n^{(i)}(l,m-1)}{b} = -f_{n-1}^{(i)}(l,m)$$

$$= \left| \mathbf{0}{}_{m}^{l}, \mathbf{1}{}_{m}^{l}, \cdots, \mathbf{N-2}{}_{m}^{l}, \mathbf{N-1}{}_{m}^{l} \right|, \quad \boldsymbol{j}{}_{m}^{l} = \begin{pmatrix} f_{n+j}^{(1)}(l,m) \\ f_{n+j}^{(2)}(l,m) \\ \vdots \\ f_{n+j}^{(N)}(l,m) \end{pmatrix}$$

$$\boldsymbol{j}_{l+1} = \boldsymbol{j}_l + a \cdot (\boldsymbol{j+1}_{l+1})$$
$$\boldsymbol{j}_{m+1} = \boldsymbol{j}_m + b \cdot (\boldsymbol{j-1}_{m+1})$$

Proposition (difference formula)

$$\tau_n(l,m) = |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \,| \qquad \tau_n(l,m+1) = |\, \mathbf{0}_{m+1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \,|$$

$$\tau_n(l+1,m) = |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}_{l+1} \,| \qquad -b\tau_n(l,m+1) = |\, \mathbf{1}_{m+1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \,|$$

$$a\tau_n(l+1,m) = |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-2}_{l+1} \,| \qquad (1+ab)\tau_n(l+1,m+1) = |\, \mathbf{0}_{m+1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}_{l+1} \,|$$

Discretization of 2DTL (3)

$$\tau_n(l,m) = |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \,|, \qquad \begin{aligned} \boldsymbol{j}_{l+1} &= \boldsymbol{j}_l + a\cdot(\boldsymbol{j+1}_{l+1}) \\ \boldsymbol{j}_{m+1} &= \boldsymbol{j}_m + b\cdot(\boldsymbol{j-1}_{m+1}) \end{aligned}$$

Verification of difference formula:

$$\begin{aligned} \tau_n(l,m) &= |\, \mathbf{0}_{l+1}, \mathbf{1}_{l+1}, \cdots, \mathbf{N-2}_{l+1}, \mathbf{N-1}_{l+1} \,| = |\, \mathbf{0} + a(\mathbf{1}_{l+1}), \mathbf{1}_{l+1}, \cdots, \mathbf{N-2}_{l+1}, \mathbf{N-1}_{l+1} \,| \\ &= |\, \mathbf{0}_l, \mathbf{1}_{l+1}, \cdots, \mathbf{N-2}_{l+1}, \mathbf{N-1}_{l+1} \,| = \cdots = |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}_{l+1} \,| \end{aligned}$$

$$\begin{aligned} a\tau_n(l+1,m) &= |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, a(\mathbf{N-1}_{l+1}) \,| = |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-2}_{l+1} - \mathbf{N-2} \,| \\ &= |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-2}_{l+1} \,| \end{aligned}$$

Plücker relation → Bilinear equation

$$\begin{aligned} 0 = \;& |\, \mathbf{0}_{m+1}, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2} \,| \times |\, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}, \mathbf{N-1}_{l+1} \,| \\ +\;& |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \,| \times |\, \mathbf{0}_{m+1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}_{l+1} \,| \\ -\;& |\, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}_{l+1} \,| \times |\, \mathbf{0}_{m+1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \,| \end{aligned}$$

→

$$\begin{aligned} 0 = \;& -b\tau_{n-1}(l,m+1) \times a\tau_{n+1}(l+1,m) \\ +\;& \tau_n(l,m) \times (1+ab)\tau_n(l+1,m+1) \\ -\;& \tau_n(l+1,m) \times \tau_n(l,m+1) \end{aligned}$$

Discretization of 2DTL (4)

Discrete 2DTL $F_n(l,m) = \dfrac{1}{ab}\log\left[1 + abe^{R_n(l,m)}\right]$

$$\Delta_{+l}\Delta_{+m}R_n(l,m) = F_{n+1}(l+1,m) + F_{n-1}(l,m+1) - F_n(l+1,m) - F_n(l,m+1)$$

Solution: $e^{R_n(l,m)} = \dfrac{\tau_{n+1}(l+1,m)\tau_{n-1}(l,m+1)}{\tau_n(l+1,m)\tau_n(l,m+1)}$ $\tau_n(l,m) = \det\left(f^{(i-1)}_{n+j-1}(l,m)\right)_{i,j=1,\ldots,N}$

$$\frac{f_n^{(i)}(l,m) - f_n^{(i)}(l-1,m)}{a} = f_{n+1}^{(i)}(l,m), \qquad \frac{f_n^{(i)}(l,m) - f_n^{(i)}(l,m-1)}{b} = -f_{n-1}^{(i)}(l,m)$$

$$f_n^{(i)}(l,m) = \alpha_i p_i^n\,(1-ap_i)^{-l}\left(1-\frac{b}{p_i}\right)^{-m} + \beta_i q_i^n\,(1-aq_i)^{-l}\left(1-\frac{b}{q_i}\right)^{-m}$$ ⟶ N-soliton solution

Remark:

- Lattice interval can be generalized to arbitrary function in corresponding independent variable, i.e., $a \to a_\ell$, $b \to b_m$

$$(1-ap_i)^{-l} \to \prod_{\nu}^{l-1}(1-a_\nu p_i)^{-1}, \quad \left(1-\frac{b}{p_i}\right)^{-m} \to \prod_{\mu}^{l-1}\left(1-\frac{b_\mu}{p_i}\right)^{-1}$$

- Discrete 1DTL is obtained by imposing $\tau_n(l+1,m+1) \doteqdot \tau_n(l,m)$ and killing m-dependence, which is realized by putting $q_i = \dfrac{1}{p_i}$

Motion of Planar Discrete Curves described by Discrete mKdV Equations

Discrete potential mKdV (I)

- **Discrete potential mKdV**

$$\tan\left(\frac{\theta_{n+1}^{m+1}-\theta_n^m}{4}\right)=\frac{b_m+a_n}{b_m-a_n}\tan\left(\frac{\theta_n^{m+1}-\theta_{n+1}^m}{4}\right)$$

- **Solution:**

$$\theta_n^m=\frac{2}{\sqrt{-1}}\log\frac{\tau_n^m}{\tau^{*m}_n}$$

$$\tau_n^m=\begin{vmatrix} f_0^{(1)} & f_1^{(2)} & \cdots & f_{N-1}^{(2)} \\ f_0^{(2)} & f_1^{(2)} & \cdots & f_{N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_0^{(N)} & f_1^{(N)} & \cdots & f_{N-1}^{(N)} \end{vmatrix}$$

$$f_s^{(i)}(n,m)=\alpha_i\ p_i^s\ (1-ap_i)^{-l}(1-bp_i)^{-m} + \beta_i\ (-p_i)^s\ (1+ap_i)^{-l}(1+bp_i)^{-m}$$

$$p_i,\ \alpha_i\in\mathbb{R},\quad \beta_i\in\sqrt{-1}\mathbb{R}$$

- **Bilinear equation:**

$$b_m\tau^{*m+1}_n\tau_{n+1}^m-a_n\tau^{*m}_{n+1}\tau_n^{m+1}+(a_n-b_m)\tau^{*m+1}_{n+1}\tau_n^m=0$$

Discrete potential mKdV (2)

Hirota-Miwa equation

(a master equation in discrete integrable systems)

$$b_m\tau_n^{m+1}(s+1)\tau_{n+1}^m(s) - a_n\tau_{n+1}^m(s+1)\tau_n^{m+1}(s) + (a_n - b_m)\tau_{n+1}^{m+1}(s+1)\tau_n^m(s) = 0$$

Solution:

$$\tau_n^m(s) = \begin{vmatrix} f_s^{(1)} & f_{s+1}^{(1)} & \cdots & f_{s+N-1}^{(1)} \\ f_s^{(2)} & f_{s+1}^{(2)} & \cdots & f_{s+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_s^{(N)} & f_{s+1}^{(N)} & \cdots & f_{s+N-1}^{(N)} \end{vmatrix}$$

$$\frac{f_s^{(i)}(n,m) - f_s^{(i)}(n-1,m)}{a_{n-1}} = f_{s+1}^{(i)}(n,m)$$

$$\frac{f_s^{(i)}(n,m) - f_s^{(i)}(n,m-1)}{b_{m-1}} = f_{s+1}^{(i)}(n,m)$$

$$f_s^{(i)} = p_i\alpha_i\, p_s^s\,(1-ap_i)^{-l}(1-bp_i)^{-m} + \beta_i\, q_s^s\,(1-aq_i)^{-l}(1-bq_i)^{-m}$$

Reduction condition: $\tau_n^m(s+1) = \text{const.} \times \tau^{*m}_n(s)$

realized by putting $q_i = -p_i, \quad p_i,\ \alpha_i \in \mathbb{R}, \quad \beta_i \in \sqrt{-1}\mathbb{R}$

$$\begin{aligned} f_{s+1}^{(i)} &= p_i\alpha_i\, p_i^s\,(1-ap_i)^{-l}(1-bp_i)^{-m} - p_i\beta_i\,(-p_i)^s\,(1+ap_i)^{-l}(1+bp_i)^{-m} \\ &= p_i\left[\alpha_i\, p_i^s\,(1-ap_i)^{-l}(1-bp_i)^{-m} - \beta_i\,(-p_i)^s\,(1+ap_i)^{-l}(1+bp_i)^{-m}\right] \\ &= p_i\,\left(f_s^{(i)}\right)^* \end{aligned}$$

Discrete potential mKdV (3)

$$b_m \tau^{*m+1}_{n} \tau^{m}_{n+1} - a_n \tau^{*m}_{n+1} \tau^{m+1}_{n} + (a_n - b_m) \tau^{*m+1}_{n+1} \tau^{m}_{n} = 0$$

divide by $\tau^{*m}_{n+1} \tau^{*m+1}_{n}$

$$b_m \frac{\tau^{m}_{n+1}}{\tau^{*m}_{n+1}} - b_m \frac{\tau^{m+1}_{n}}{\tau^{*m+1}_{n}} = -(a_n - b_m) \frac{\tau^{*m+1}_{n+1} \tau^{m}_{n}}{\tau^{*m}_{n+1} \tau^{*m+1}_{n}}, \qquad \frac{\tau^{m}_{n}}{\tau^{*m}_{n}} = e^{\sqrt{-1}\theta^m_n/2}$$

$$\longrightarrow \boxed{b_m e^{\sqrt{-1}\theta^m_{n+1}/2} - a_n e^{\sqrt{-1}\theta^{m+1}_n/2} = -(a_n - b_n) \frac{\tau^{*m+1}_{n+1} \tau^{m}_{n}}{\tau^{*m+1}_{n} \tau^{*m}_{n+1}}}$$

complex conjugate：

$$\longrightarrow \boxed{b_m e^{\sqrt{-1}\theta^{m+1}_n/2} - a_n e^{\sqrt{-1}\theta^m_{n+1}/2} = -(a_n - b_n) \frac{\tau^{m+1}_{n+1} \tau^{*m}_{n}}{\tau^{*m+1}_{n} \tau^{*m}_{n+1}}}$$

$$\frac{b_m e^{\sqrt{-1}\theta^{m+1}_n/2} - a_n e^{\sqrt{-1}\theta^m_{n+1}/2}}{b_m e^{\sqrt{-1}\theta^m_{n+1}/2} - a_n e^{\sqrt{-1}\theta^{m+1}_n/2}} = e^{\sqrt{-1}(\theta^{m+1}_{n+1} - \theta^m_n)/2} \longleftrightarrow$$

discrete pmKdV

Motion of discrete curves and preceding works

- **Purpose: formulation of discrete motions of plane discrete curves described by discrete mKdV.**
- **Preceding works**
  - Curves and solitons

    Hasimoto (1972): Vortex filament and NLS

    Lamb (1976): space curve and NLS, mKdV

    Goldstein-Petrich (1991): planar curve and mKdV
  - Continuous motion of discrete curves

    Hisakado-Nakayama-Wadati (1995): a semi-discrete mKdV

    Doliwa-Santini (1995): dynamics on 3-sphere ($R=1/\lambda$, $\lambda\to 0$: plane curve)

    Hoffmann-Kutz (2004): semi-discrete mKdV

    Pinkall-Springborn-Weissmann (2007): discrete NLS
  - Discrete motion of discrete curves

    Doliwa-Santini (1999): discrete sine-Gordon on 3-sphere

    Fujioka-Kurose(2007): discrete Burgers on complex hyperbola

    Inoguchi-K-Matsuura-Ohta (2010) : discrete mKdV on Euclidean plane

Discrete motion of planar discrete curve (I)

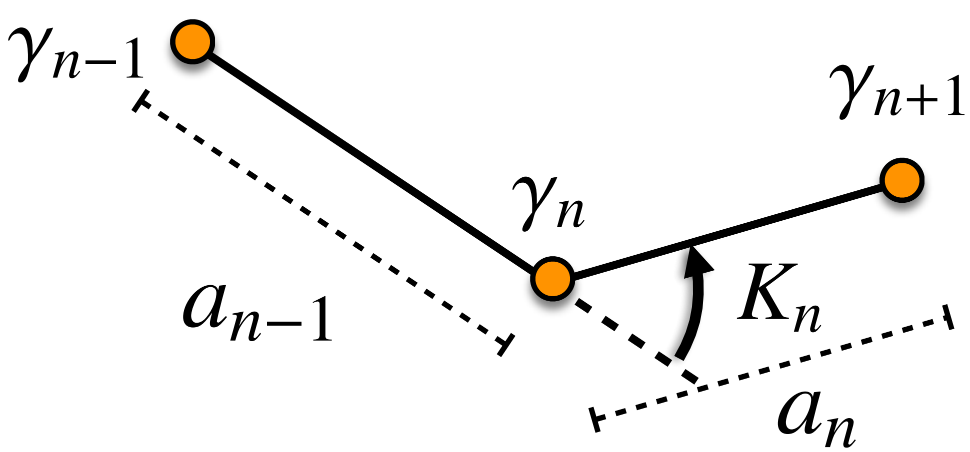
Smooth curve:

$$|\gamma'| = 1, \quad \gamma' = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \kappa = \theta'$$

$$\frac{\partial}{\partial s}\gamma' = \begin{bmatrix} 0 & -\kappa \\ -\kappa & 0 \end{bmatrix}\gamma', \quad \frac{\partial}{\partial t}\gamma' = \begin{bmatrix} 0 & \kappa_{ss} + \frac{\kappa^3}{2} \\ -\kappa_{ss} - \frac{\kappa^3}{2} & 0 \end{bmatrix}\gamma' \qquad \theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0$$

Discrete curve:

Def. $\gamma : \mathbb{Z} \to \mathbb{R}^2;\ n \to \gamma_n$ planar discrete curve $\longleftrightarrow$ $\left|\frac{\gamma_{n+1} - \gamma_n}{a_n}\right| = 1$



$$\frac{\gamma_{n+1} - \gamma_n}{a_n} = \begin{bmatrix} \cos\Psi_n \\ \sin\Psi_n \end{bmatrix}$$

Ψ_n : turning angle

Discrete Frenet formula

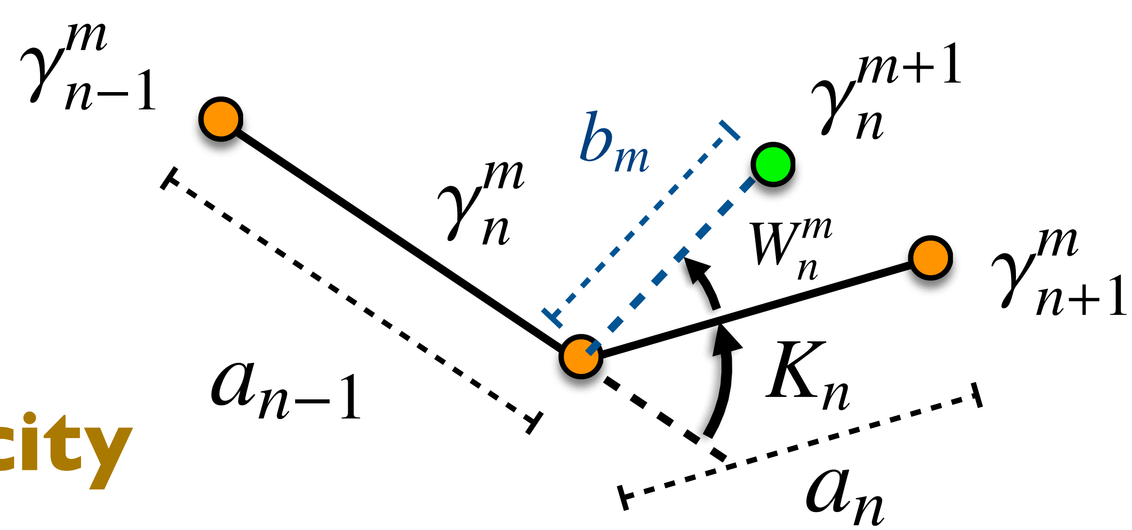
$$\frac{\gamma_{n+1} - \gamma_n}{a_n} = R(K_n)\,\frac{\gamma_n - \gamma_{n-1}}{a_{n-1}}, \qquad R(K_n) = \begin{bmatrix} \cos K_n & -\sin K_n \\ \sin K_n & \cos K_n \end{bmatrix}$$

Discrete motion of planar discrete curve (2)

- **Discrete Frenet formula:**

$$\frac{\gamma_{n+1}-\gamma_n}{a_n} = R(K_n)\,\frac{\gamma_n-\gamma_{n-1}}{a_{n-1}},\qquad R(K_n) = \begin{bmatrix} \cos K_n & -\sin K_n \\ \sin K_n & \cos K_n \end{bmatrix}$$

- **Discrete motion:**



$$\frac{\gamma_n^{m+1}-\gamma_n^m}{b_m} = R(W_n^m)\,\frac{\gamma_{n+1}^m-\gamma_n^m}{a_n}$$

- **Compatibility + isoperimetricity**

$$(\gamma_{n+1})^{m+1} = \left(\gamma^{m+1}\right)_{n+1},\qquad \left|\frac{\gamma_{n+1}^{m+1}-\gamma_n^{m+1}}{a_n}\right| = 1$$

$$\longrightarrow\quad K_n^{m+1}-K_{n+1}^m = W_{n+1}^m - W_{n-1}^m,\qquad \tan\left(\frac{W_{n+1}^m+K_{n+1}^m}{2}\right) = \frac{b_m+a_n}{b_m-a_n}\tan\frac{W_n^m}{2}$$

- **Potential function and discrete potential mKdV:**

$$W_m^n = \frac{\theta_n^{m+1}-\theta_{n+1}^m}{2},\qquad K_n^m = \frac{\theta_{n+1}^m-\theta_{n-1}^m}{2}\qquad \Psi_n^m = \frac{\theta_{n+1}^m+\theta_n^m}{2}$$

$$\longrightarrow\qquad \tan\left(\frac{\theta_{n+1}^{m+1}-\theta_n^m}{4}\right) = \frac{b_m+a_n}{b_m-a_n}\tan\left(\frac{\theta_n^{m+1}-\theta_{n+1}^m}{4}\right)$$

Explicit formula

- **Explicit formula for $\gamma_n^m \in \mathbb{R}^2$ in terms of τ function**

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = R\left(\frac{\theta_{n+1}^m - \theta_{n-1}^m}{2}\right) \frac{\gamma_n^m - \gamma_{n-1}^m}{a_{n-1}} \qquad \frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = R\left(\frac{\theta_n^{m+1} - \theta_{n+1}^m}{2}\right) \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}$$

$$\tan\left(\frac{\theta_{n+1}^{m+1} - \theta_n^m}{4}\right) = \frac{b_m + a_n}{b_m - a_n} \tan\left(\frac{\theta_n^{m+1} - \theta_{n+1}^m}{4}\right)$$

Proposition Let τ_n^m be a solution to the following bilinear equations:

$$b_m \tau^{*m+1}_{n} \tau^m_{n+1} - a_n \tau^{*m}_{n+1} \tau^{m+1}_{n} + (a_n - b_n) \tau^{*m+1}_{n+1} \tau^m_n = 0$$

$$D_y\, \tau^m_{n+1} \cdot \tau^m_n = -a_n \tau^{*m}_{n+1} \tau^{*m}_{n}, \qquad D_y\, \tau^{m+1}_n \cdot \tau^m_n = -b_m \tau^{*m+1}_{n} \tau^{*m}_{n},$$

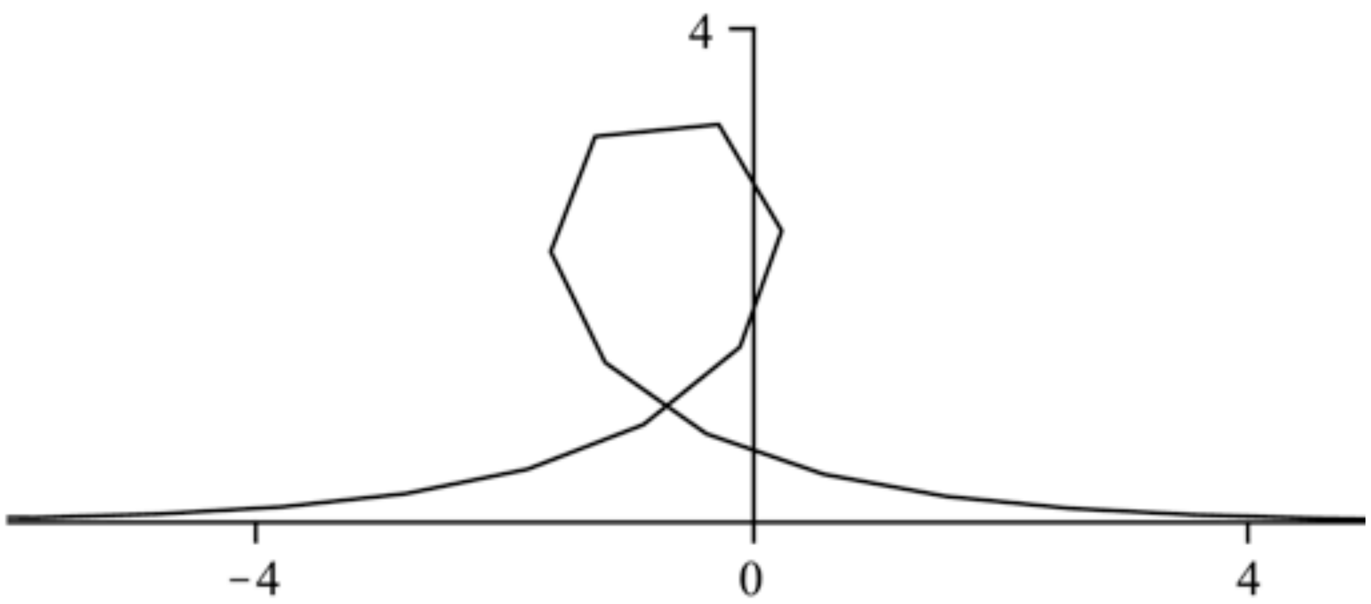
Then,

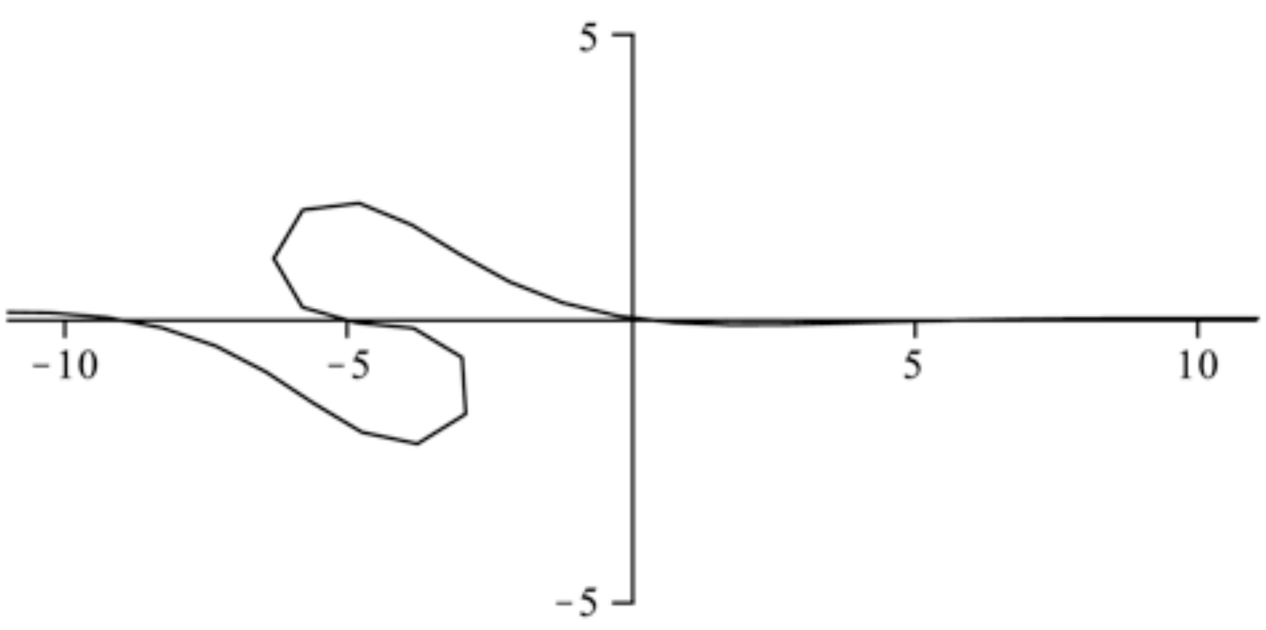
$$\Theta_n^m = \frac{2}{\sqrt{-1}} \log \frac{\tau_n^m}{\tau^{*m}_n}, \qquad \gamma_n^m = \begin{pmatrix} -\dfrac{1}{2} (\log \tau_n^m \tau^{*m}_n)_y \\ \dfrac{1}{2\sqrt{-1}} \left(\log \dfrac{\tau_n^m}{\tau^{*m}_n}\right)_y \end{pmatrix}$$

Discrete Motion of Planar Discrete Curve (4)

$$\tau_n^m = e^{-\left(\sum^{n-1} a_i + \sum^{m-1} b_j\right)y} \ \det\left(f_{j-1}^{(i)}\right)_{i,j=1,\ldots,N}$$

$$f_s^{(i)} = \alpha_i p_i^s e^{\frac{1}{p_i}y} \prod_{l_1}^{n-1}(1-a_{l_1}p_i)^{-1} \prod_{l_2}^{m-1}(1-b_{l_2}p_i)^{-1} + \beta_i(-p_i)^s e^{-\frac{1}{p_i}y} \prod_{l_1}^{n-1}(1+a_{l_1}p_i)^{-1} \prod_{l_2}^{m-1}(1+b_{l_2}p_i)^{-1}$$

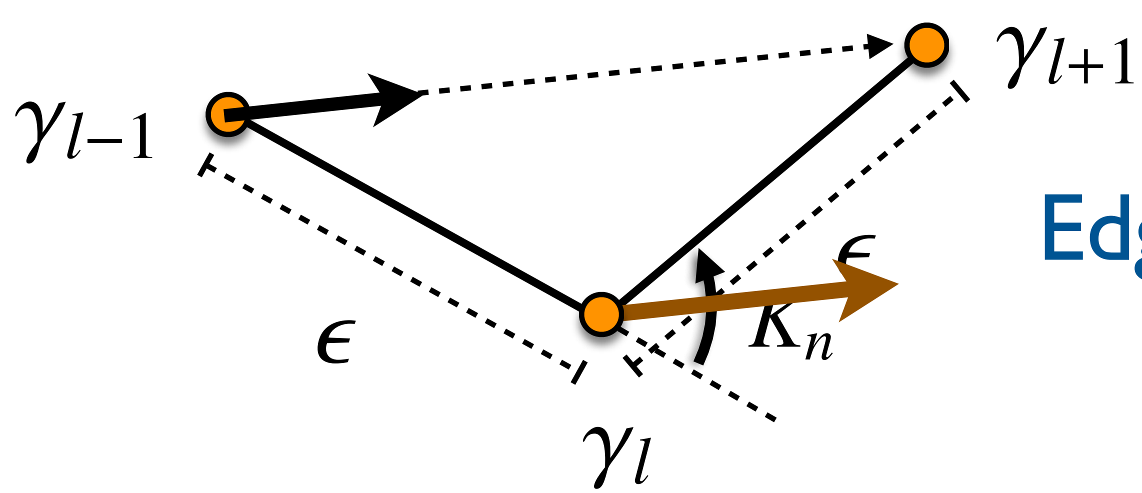




Continuous motion of discrete curve (I)

Edge tangential flow of discrete curve

A.Doliwa and P.M. Santini (1994), T. Hoffmann and N. Kutz (2004)



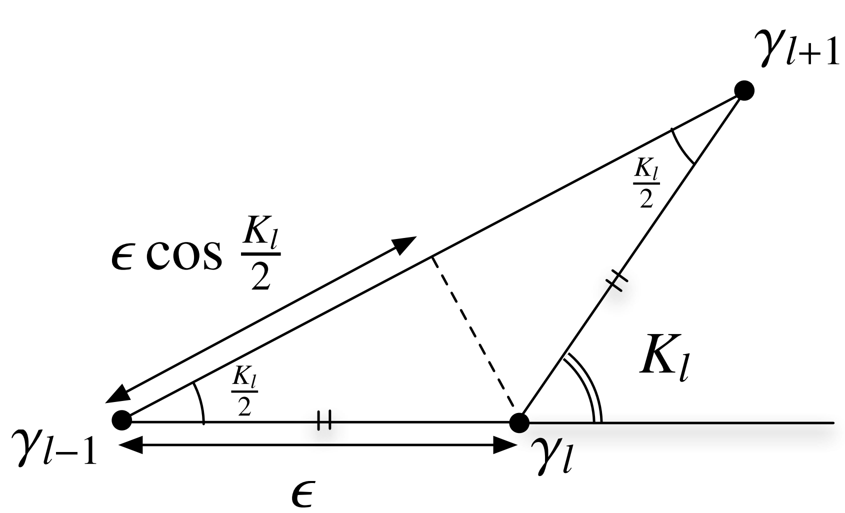
$$\frac{\gamma_{l+1}-\gamma_l}{\epsilon} = R(K_l)\,\frac{\gamma_l-\gamma_{l-1}}{\epsilon}$$

Edge tangential vector

$$\Delta^h \gamma_l := 2\epsilon\,\frac{\gamma_{l+1}-\gamma_{l-1}}{\|\,\gamma_{l+1}-\gamma_{l-1}\|^2}$$

Edge tangential flow:

$$\frac{d\gamma_l}{d\zeta} = \alpha\Delta^h\gamma_l = 2\epsilon\alpha\,\frac{\gamma_{l+1}-\gamma_{l-1}}{\|\,\gamma_{l+1}-\gamma_{l-1}\|^2}$$

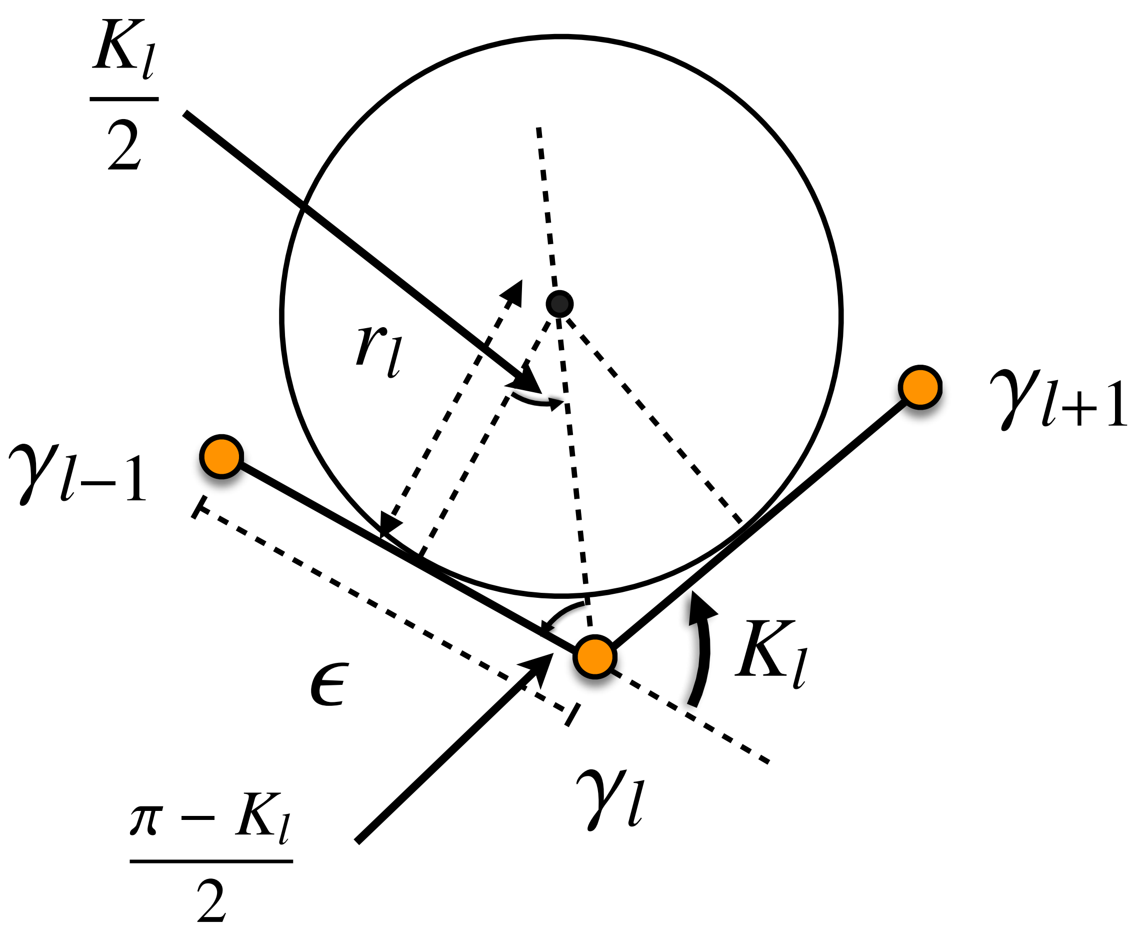


$$\|\gamma_{l+1}-\gamma_{l-1}\| = 2\epsilon\cos\frac{K_l}{2},\quad \gamma_{l+1}-\gamma_{l-1} = 2\cos\frac{K_l}{2}\,R\left(-\frac{K_l}{2}\right)(\gamma_{l+1}-\gamma_l)$$

$$\downarrow$$

$$\frac{d\gamma_l}{d\zeta} = \frac{\alpha}{\cos\frac{K_l}{2}}\,R\left(-\frac{K_l}{2}\right)\frac{\gamma_{l+1}-\gamma_l}{\epsilon}$$

Continuous motion of discrete curve (2)



$$\frac{\gamma_{l+1} - \gamma_l}{\epsilon} = R(K_l)\,\frac{\gamma_l - \gamma_{l-1}}{\epsilon}$$

$$\frac{d\gamma_l}{d\zeta} = \frac{\alpha}{\cos\frac{K_l}{2}}\,R\left(-\frac{K_l}{2}\right)\frac{\gamma_{l+1} - \gamma_l}{\epsilon}$$

$$\frac{\gamma_{l+1} - \gamma_l}{\epsilon} = \begin{bmatrix} \cos\Psi_l \\ \sin\Psi_l \end{bmatrix}$$

- **Compatibility**： **semi-discrete potential mKdV**

$$\Psi_l = \frac{\theta_{l+1} + \theta_l}{2}, \quad K_l = \frac{\theta_{l+1} - \theta_{l-1}}{2}$$

$$\frac{d}{d\zeta}\theta_l = \frac{2}{\epsilon}\tan\left(\frac{\theta_{l+1} - \theta_{l-1}}{4}\right)$$

- **"Discrete curvature" → semi-discrete mKdV**

$$\kappa_l := \frac{1}{r_l} = \frac{2}{\epsilon}\tan\frac{K_l}{2} = \frac{d\theta_l}{d\zeta} \longrightarrow \frac{d\kappa_l}{d\zeta} = \frac{2}{\epsilon}\left(1 + \frac{\epsilon^2}{4}\kappa_l^2\right)(\kappa_{l+1} - \kappa_{l-1})$$

Continuous motion of discrete curve (3)

Soliton solutions

$$\tau_l(s;y) = e^{(-s+\epsilon l)y}\begin{vmatrix} f_0^{(1)} & f_1^{(2)} & \cdots & f_{N-1}^{(2)} \\ f_0^{(2)} & f_1^{(2)} & \cdots & f_{N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_0^{(N)} & f_1^{(N)} & \cdots & f_{N-1}^{(N)} \end{vmatrix}$$

$$f_n^{(i)} = \alpha_i p_i^n (1-\epsilon p_i)^{-l} e^{\eta_i} + \beta_i(-p_i)^n(1+\epsilon p_i)^{-l} e^{\xi_i},$$

$$\eta_i = \frac{p_i}{1-\epsilon^2 p_i^2}s + \frac{1}{p_i}y, \quad \xi_i = -\frac{p_i}{1-\epsilon^2 p_i^2}s - \frac{1}{p_i}y,$$

$$\alpha_i,\ p_i \in \mathbb{R}, \quad \beta_i \in \sqrt{-1}\mathbb{R}$$

$$\theta_l = \frac{2}{\sqrt{-1}}\log\frac{\tau_l}{\tau_l^*}, \quad \gamma_l = \begin{bmatrix} -\frac{1}{2}\left(\log \tau_l\tau_l^*\right)_y \\ \frac{1}{2\sqrt{-1}}\left(\log\frac{\tau_l}{\tau_l^*}\right)_y \end{bmatrix}$$

Bilinear equations

semi-discrete mKdV

$$D_s\, \tau_l \cdot \tau_l^* = \frac{1}{2\epsilon}\left(\tau_{l-1}^*\tau_{l+1} - \tau_{l+1}^*\tau_{l-1}\right),$$

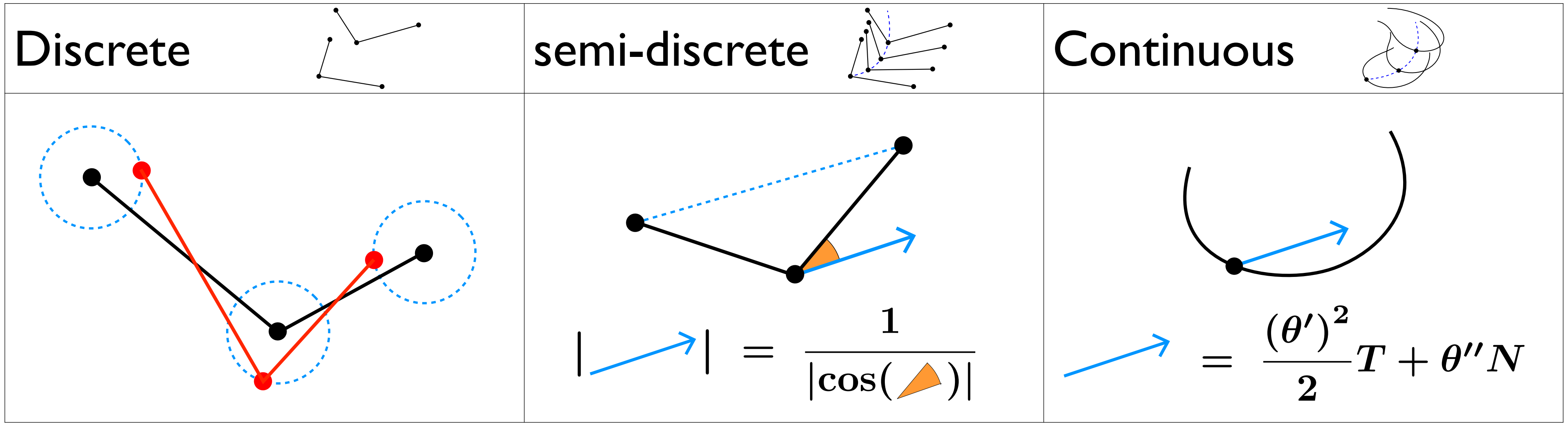
$$\tau_l\tau_l^* = \frac{1}{2}\left(\tau_{l-1}^*\tau_{l+1} + \tau_{l+1}^*\tau_{l-1}\right)$$

motion of curves

$$\frac{1}{2}D_sD_y\, \tau_l \cdot \tau_l = \tau_l^2 - \tau_{l+1}^*\tau_{l-1}^*,$$

$$D_y\, \tau_{l+1}\cdot\tau_l = \epsilon\tau_{l+1}\tau_l - \epsilon\tau_{l+1}^*\tau_l^*$$

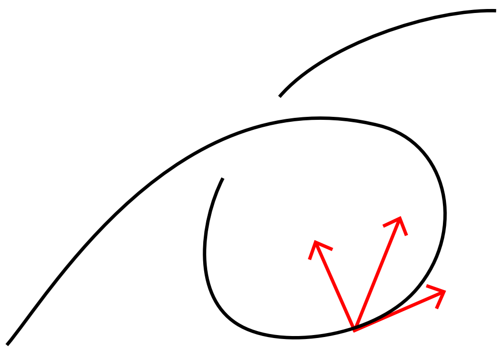
Summary



- Isoperimetric motions are described by potential mKdV equations
  - **Discrete:**
    $$\tan\left(\frac{\theta_{n+1}^{m+1} - \theta_n^m}{4}\right) = \frac{b_m + a_n}{b_m - a_n} \tan\left(\frac{\theta_n^{m+1} - \theta_{n+1}^m}{4}\right)$$
    $$\arg \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = \frac{\theta_{n+1}^m + \theta_n^m}{2}, \quad a_n = |\gamma_{n+1}^m - \gamma_n^m|, \quad b_m = |\gamma_n^{m+1} - \gamma_n^m|$$
  - **Semi-discrete:**
    $$\frac{d}{d\zeta}\theta_l = \frac{2}{\epsilon} \tan\left(\frac{\theta_{l+1} - \theta_{l-1}}{4}\right)$$
    $$\arg \frac{\gamma_{l+1} - \gamma_l}{\epsilon} = \frac{\theta_{l+1} + \theta_l}{2}, \quad \epsilon = |\gamma_{l+1} - \gamma_l|$$
  - **Continuous:**
    $$\theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0 \qquad \arg\gamma' = \theta$$

Motion of Space Curves described by mKdV Equations (optional)

Motion of smooth space curves



arc-length parametrized $|\gamma'| = 1$ tangent vector $T := \gamma'$

normal vector $N := \dfrac{T'}{|T|}$ binormal vector $B := T \times N$

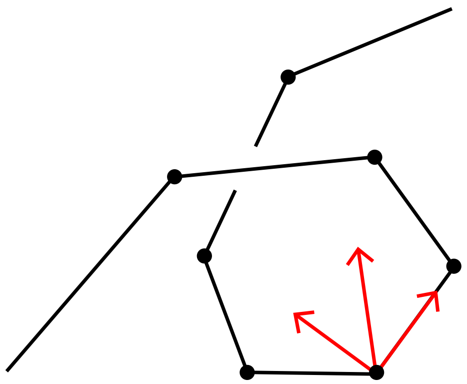
Frenet frame $F := (T, N, B)$

- **Frenet-Serret formula:**

$$F' = F \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \lambda \\ 0 & -\lambda & 0 \end{bmatrix}$$

curvature $\kappa = |T'|$ torsion $\lambda = -\langle N, B' \rangle$

Space discrete curve (I)

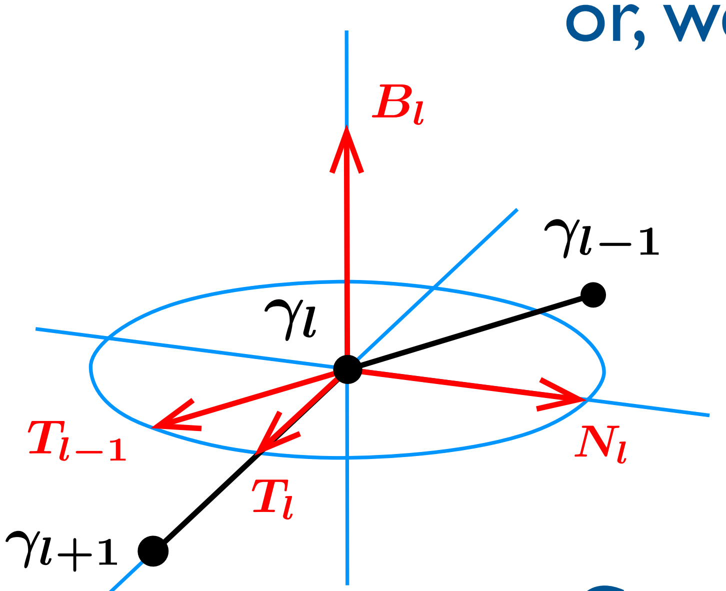


$$\epsilon_l := |\gamma_{l+1} - \gamma_l|$$

tangent vector $$T_l := \frac{\gamma_{l+1} - \gamma_l}{\epsilon_l}$$

binormal vector $$B_l := \frac{T_{l-1} \times T_l}{|T_{l-1} \times T_l|}$$

normal vector $$N_l := B_l \times T_l$$



or, we may define as

$$N_l := \frac{\Delta T_l - \langle \Delta T_l, T_l \rangle T_l}{|\Delta T_l - \langle \Delta T_l, T_l \rangle T_l|}, \quad \Delta T_l := \frac{T_l - T_{l-1}}{\epsilon_l + \epsilon_{l-1}}$$

$$B_l := T_l \times N_l$$

Crucial point of the above definition is to choose as

$$N_l \in \mathrm{span}\,\{T_l,\ T_{l-1}\}$$

Frenet frame: $F_l := (T_l, N_l, B_l)$

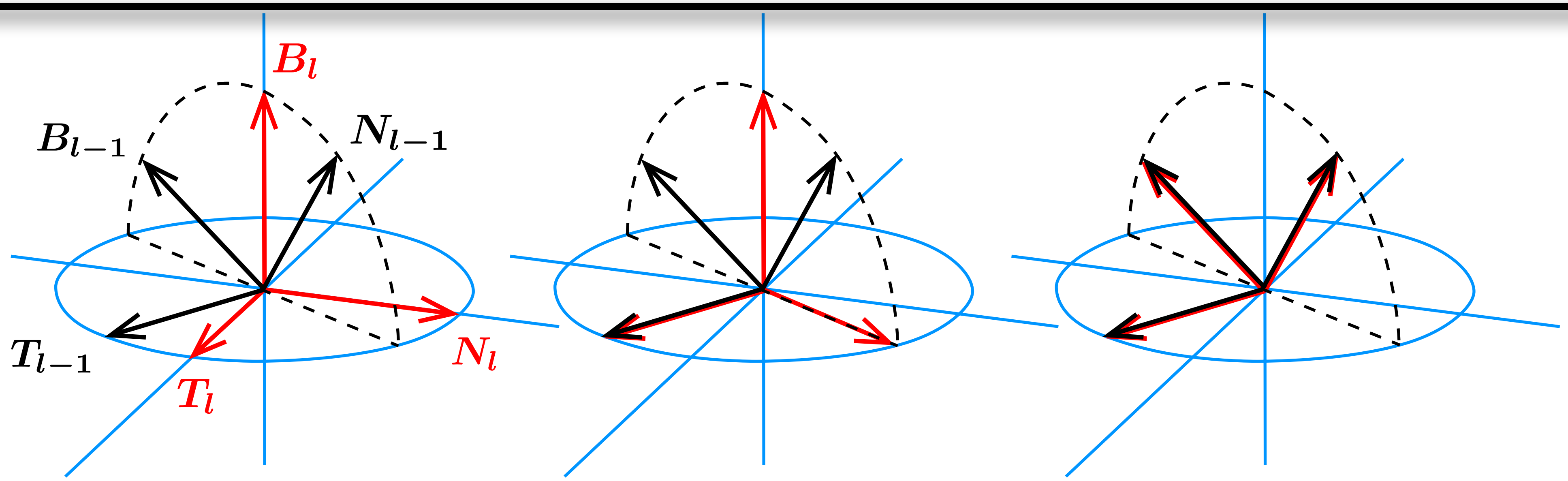
Space discrete curve (2)

$$F_l = (T_l, N_l, B_l), \quad B_l = \frac{T_{l-1} \times T_l}{|T_{l-1} \times T_l|}, \quad N_l = B_l \times T_l$$

Discrete Frenet-Serret formula:

$$F_{l-1} = F_l \begin{pmatrix} \cos\kappa_l & \sin\kappa_l & 0 \\ -\sin\kappa_l & \cos\kappa_l & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_l & -\sin\lambda_l \\ 0 & \sin\lambda_l & \cos\lambda_l \end{pmatrix}$$

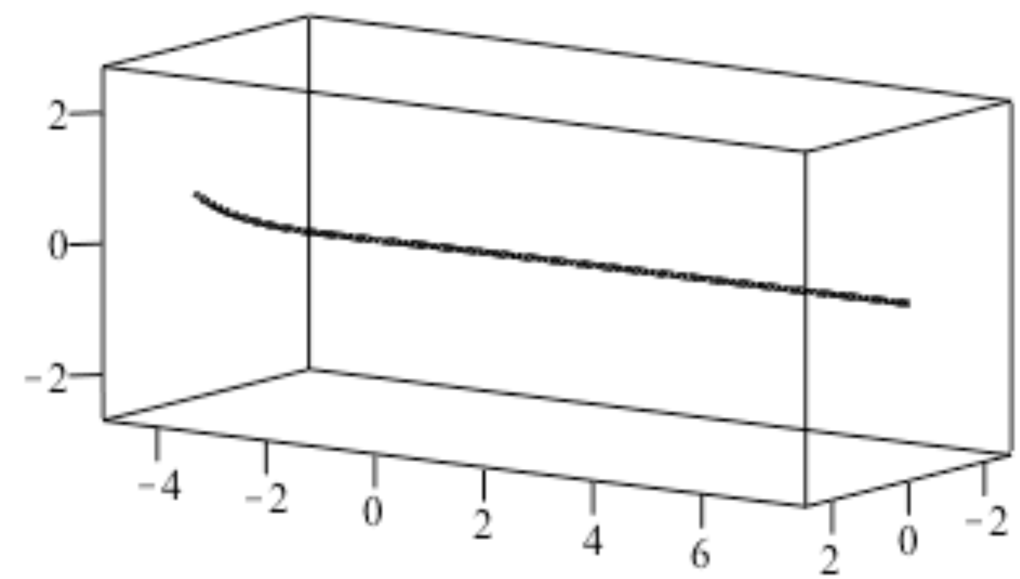
$$\langle T_l, T_{l-1} \rangle = \cos\kappa_l, \quad \langle B_l, B_{l-1} \rangle = \cos\lambda_l, \quad \langle B_l, N_{l-1} \rangle = \sin\lambda_l$$



Motion of space curves

Smooth curve:

$\gamma = \gamma(s,t)$ torsion $\lambda = \text{const.}$



Motion: $$\dot{\gamma} = \left(\frac{\kappa^2}{2} - 3\lambda^2\right) T + \kappa' N - 2\lambda\kappa B$$

→ mKdV: $$\dot{\kappa} = \kappa''' + \frac{3}{2}\kappa^2\kappa'$$

continuous motion of discrete curve:

$\gamma_l = \gamma_l(\zeta)$ $\epsilon_l = |\gamma_{l+1} - \gamma_l| = \text{const.}$, $\lambda_l = \angle(B_l, B_{l-1}) = \text{const.}$

Motion: $$\dot{\gamma}_l = \cos\lambda\, T_l - \cos\lambda \tan\frac{\kappa_l}{2}\, N_l + \sin\lambda\tan\frac{\kappa_l}{2}\, B_l$$

→ semi-discrete mKdV: $$\dot{\kappa}_l = \frac{1}{\epsilon}\left(\tan\frac{\kappa_{l+1}}{2} - \tan\frac{\kappa_{l-1}}{2}\right)$$ $\kappa_l = \angle(T_l, T_{l-1})$

Summary & Future Problems

- **Construction of a model of motion of plane discrete curve described by discrete (potential) mKdVs**
    - Casorati determinant solution to pmKdV (solitons, breathers)
    - Explicit formula of discrete curve in terms of the τ function.
    - Bäcklund transformations
- **Discretization preserving integrability:** determinant structure of τ function
- **Curve motions in various settings: generalization to space curves**
- **Combine elasticity to discrete curve dynamics (Nishinari, 2001)**