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Introduction to Integrable Systems
―based on two-dimensional Toda lattice―

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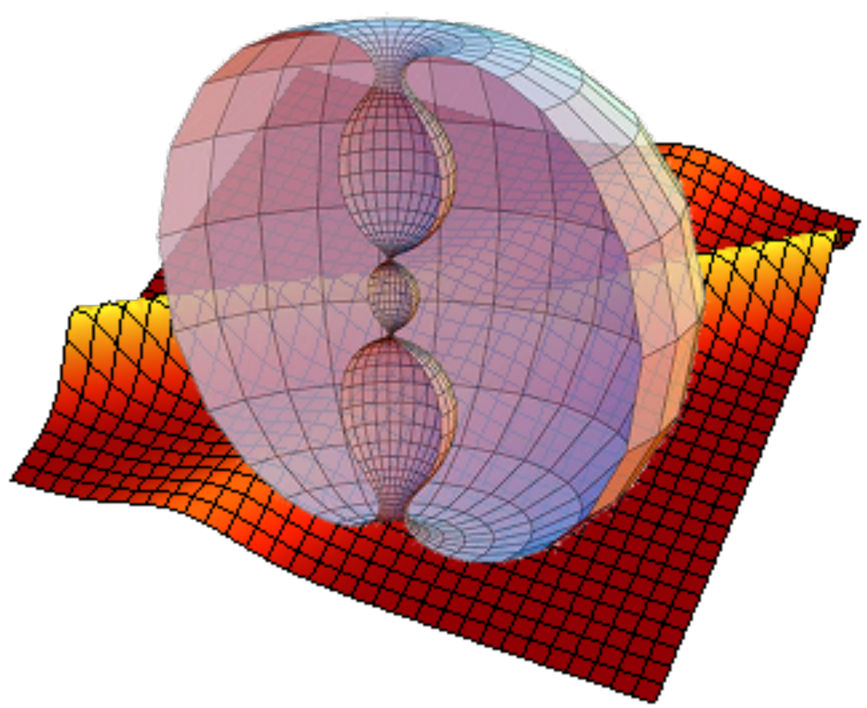
Chapter 1

First step of integrable systems begins from the Toda lattice



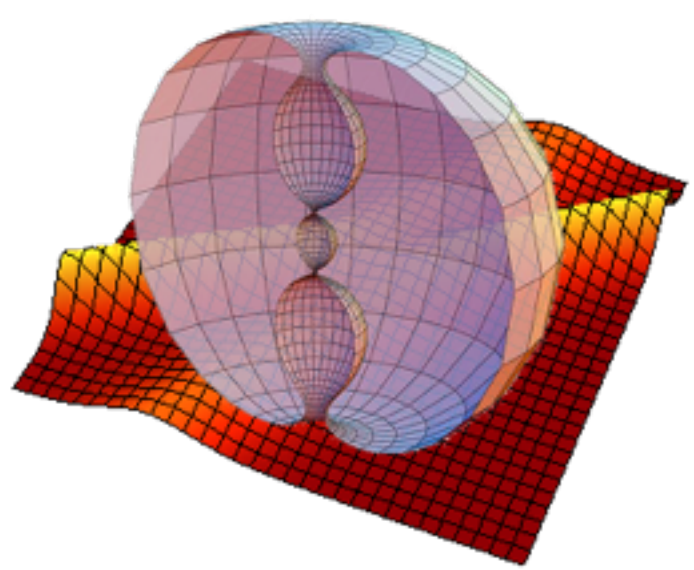
Contents and Keywords

Toda Lattice and Its Basic Properties

- **Completely Integrable Systems**
- **Lax Formalism：**

  compatibility of linear systems

- **Bäcklund transformation**
- **Hirota's method：Construct exact solutions**
- **Soliton solutions**



Toda Lattice Equation

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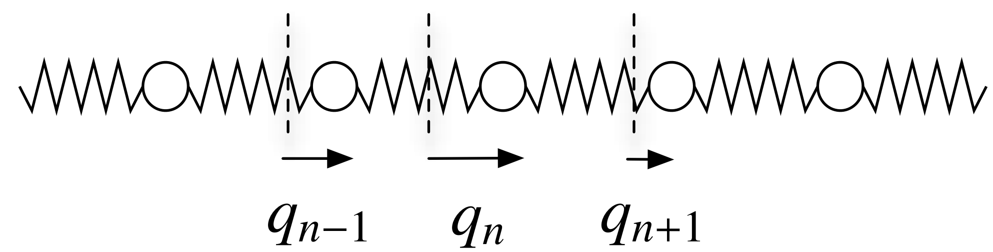
Toda Lattice Eq. $$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$$

Relative displacement: $r_n = q_n - q_{n-1}$

Potential energy: $\phi(r) \Longrightarrow$ Force $-\phi'(r)$



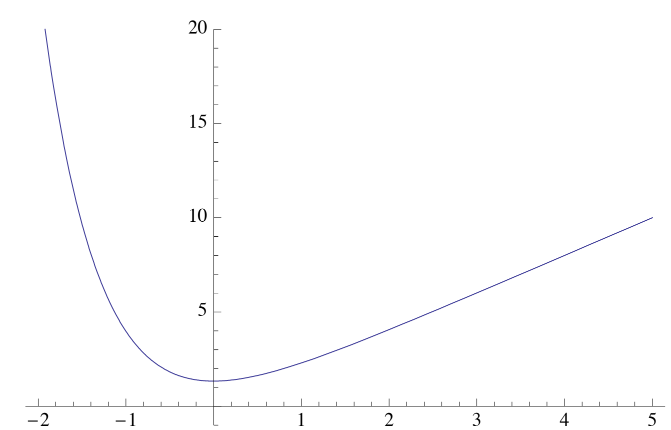
Equation of motion $$m\frac{d^2 q_n}{dt^2} = -\phi'(r_n) + \phi'(r_{n+1})$$

- **Hooke's law:** $\phi(r) = \frac{1}{2}\kappa r^2$ Force: $-\phi'(r) = -\kappa r$

Equation of motion: $$\frac{d^2 q_n}{dt^2} = -\kappa(q_n - q_{n-1}) + \kappa(q_{n+1} - q_n) = \kappa(q_{n+1} + q_{n-1} - 2q_n)$$

- **Toda potential:** $\phi(r) = \frac{a}{b}e^{-br} + ar \quad a, b > 0$ Force: $-\phi'(r) = a(e^{-br} - 1)$

Rem. $r \sim 0: \quad \phi(r) \sim \frac{a}{b} + \frac{ab}{2}r^2$ **Hooke's law**



Equation of motion: $$m\frac{d^2 q_n}{dt^2} = a\left[e^{-b(q_n - q_{n-1})} - e^{-b(q_{n+1} - q_n)}\right]$$

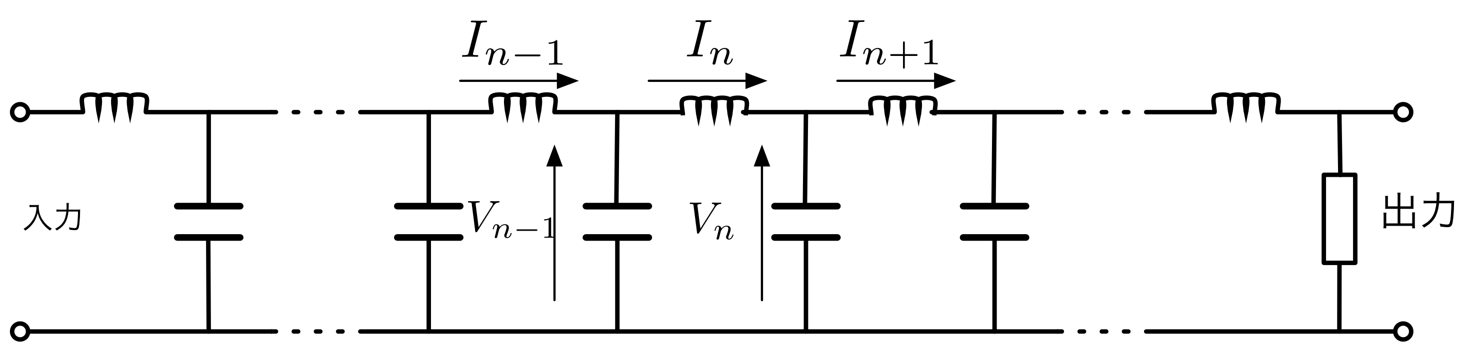
Various Forms of Toda Lattice

Toda Lattice Eq. $$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$$

$$\frac{d^2 r_n}{dt^2} = e^{r_{n+1}} + e^{r_{n-1}} - 2e^{r_n} \qquad r_n := q_n - q_{n+1}$$

$$\frac{d^2}{dt^2}\log(1+V_n) = V_{n+1} + V_{n-1} - 2V_n \quad \textbf{or} \quad \begin{cases} \dfrac{d}{dt}\log(1+V_n) = I_n - I_{n+1}, \\ \dfrac{dI_n}{dt} = V_{n-1} - V_n \end{cases}$$

$$\begin{cases} 1 + V_n = e^{r_n}, \\ I_n = \dfrac{dq_n}{dt} \end{cases}$$



$$\begin{cases} \dfrac{da_n}{dt} = a_n(b_n - b_{n+1}), \\ \dfrac{db_n}{dt} = 2(a_{n-1}^2 - a_n^2) \end{cases} \qquad a_n = \frac{1}{2}e^{\frac{q_n - q_{n+1}}{2}}, \quad b_n = \frac{1}{2}\frac{dq_n}{dt}$$

Properties of Toda Lattice (I)

- Hamilton system of classical mechanics, with the Hamiltonian

$$H = \frac{1}{2m}\sum_n p_n^2 + \frac{a}{b}\sum_n e^{-b(q_n - q_{n-1})}, \quad q_n = q_n, \quad p_n = m\frac{dq_n}{dt},$$

- In the case of finite system with N particles (e.g. periodic system), there are N conserved quantities commuting w.r.t. the Poisson bracket. Namely, it is completely integrable systems, and the initial value problem can be solved by quadrature.

Liouville-Arnold's Theorem:
If a Hamilton system with N degrees of freedom possesses N conserved quantities commuting w.r.t. the Poisson bracket, then the initial value problem is solved by finite times applications of quadrature, namely,

- arithmetic operations
- differentiation & integration
- taking inverse function
- solving equations without differentiation

- Examples of completely integrable systems:
  - 2-body problem (Kepler problem)
  - Lagrange top, Euler top, Kowalevskaya top (1888)
  - Toda lattice (M. Toda, 1967)

Properties of Toda Lattice (2)

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- Formulation as the spectral preserving deformation of an eigenvalue problem of a linear operator (Lax formalism):

$$\frac{dI_n}{dt} = V_{n-1} - V_n, \quad \frac{d}{dt}\log(1+V_n) = I_n - I_{n+1}, \quad n = 1,\ldots,N, \quad I_{N+1} = I_1, \quad V_{N+1} = V_1$$

$$L\Psi = \lambda\Psi, \quad L = \begin{pmatrix} I_1 & 1 & & & & 1+V_N \\ 1+V_1 & I_2 & 1 & & & \\ & 1+V_2 & I_3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1+V_{N-2} & I_{N-1} & 1 \\ 1 & & & & 1+V_{N-1} & I_N \end{pmatrix}$$

$$\frac{d\Psi}{dt} = B\Psi, \quad B = \begin{pmatrix} 0 & & & & 1+V_N \\ 1+V_1 & 0 & & & \\ & 1+V_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1+V_{N-2} & 0 \\ & & & & 1+V_{N-1} & 0 \end{pmatrix}$$

- Compatibility condition with $\lambda_t = 0$:

$$\frac{dL}{dt}\Psi + L\frac{d\Psi}{dt} = \lambda\frac{d\Psi}{dt} \rightarrow \frac{dL}{dt}\Psi + LB\Psi = BL\Psi \rightarrow \boxed{\frac{dL}{dt} = BL - LB} \Rightarrow \text{Toda Lattice}$$

A Merit of Lax Formalism

A merit of Lax formalism： construction of conserved quantities

Prop： Tr L^k (k=1,...N) are conserved quantities.

$$\frac{d}{dt}\mathrm{Tr}\, L^k = 0, \quad k = 1, \ldots, N.$$

証明： **For** $A = (a_{ij}), \quad B = (b_{ij})$ $\qquad \mathrm{Tr}\, A = \sum_{i=1}^N a_{ii}, \quad \mathrm{Tr}\, AB = \sum_{i=1}^N \sum_{k=1}^N a_{ik} b_{ki} = \sum_{k=1}^N \sum_{i=1}^N b_{ki} a_{ik} = \mathrm{Tr}\, BA$

If entries of A and B are functions of t, then

$$\frac{d}{dt}\mathrm{Tr}\, AB = \frac{d}{dt}\sum_{i=1}^N \sum_{k=1}^N a_{ik} b_{ki} = \sum_{i=1}^N \sum_{k=1}^N \left(a'_{ik} b_{ki} + a_{ik} b'_{ki}\right) = \mathrm{Tr}\left(\frac{dA}{dt}B + A\frac{dB}{dt}\right).$$

Since L’ = BL - LB, we have

$$\frac{d}{dt}\mathrm{Tr}\, L^k = \mathrm{Tr}\left(L'L^{k-1} + LL'L^{k-2} + \cdots + L^{k-1}L'\right) = \mathrm{Tr}\left[(BL-LB)L^{k-1} + L(BL-LB)L^{k-2} + \cdots + L^{k-1}(BL-LB)\right]$$

$$= \mathrm{Tr}\left[(BL^k - LBL^{k-1}) + (LBL^{k-1} - LBL^{k-2}) + \cdots + (L^{k-1}BL - L^kB)\right] = \mathrm{Tr}\left(BL^k - L^kB\right) = 0. \qquad \square$$

※ **In terms of q_n Tr L ∝ momentum (運動量)， $\mathrm{Tr}L^2$ ∝ total energy**

Bäcklund Transformation

$$(*)\begin{cases} \dfrac{dq_n}{dt} = \lambda e^{q_n-\overline{q}_n} + \dfrac{1}{\lambda} e^{\overline{q}_{n-1}-q_n} + \alpha \\ \dfrac{d\overline{q}_n}{dt} = \lambda e^{q_n-\overline{q}_n} + \dfrac{1}{\lambda} e^{\overline{q}_n-q_{n+1}} + \alpha \end{cases}$$

→ eliminate $\overline{q}_n(q_n)$ →

$$\begin{cases} \dfrac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \\ \dfrac{d^2 \overline{q}_n}{dt^2} = e^{\overline{q}_{n-1}-\overline{q}_n} - e^{\overline{q}_n-\overline{q}_{n+1}} \end{cases}$$

Solving (*) for given q_n, we obtain another solution $\overline{q}_n$: Bäcklund transformation

- **Example:** $q_n = 0,\ \lambda = e^{-\kappa},\ \alpha = -(e^{\kappa} + e^{-\kappa})$

$$\begin{cases} 0 = \lambda e^{-\overline{q}_n} + \dfrac{1}{\lambda} e^{\overline{q}_{n-1}} + \alpha \\ \dfrac{d\overline{q}_n}{dt} = \lambda e^{-\overline{q}_n} + \dfrac{1}{\lambda} e^{\overline{q}_n} + \alpha \end{cases} \rightarrow e^{\overline{q}_n} = X_n \rightarrow \begin{cases} X_n = -\dfrac{e^{-\kappa}}{e^{\kappa}X_{n-1} - (e^{\kappa}+e^{-\kappa})} & \text{discrete Riccati eq.} \\ X_n' = e^{\kappa}X_n^2 - (e^{\kappa}+e^{-\kappa})X_n + e^{-\kappa} & \text{Riccati eq.} \end{cases}$$

→

$$X_n = \frac{1 + e^{2\kappa(n-1)+2\beta t}}{1 + e^{2\kappa n + 2\beta t}}, \quad \beta = \sinh\kappa = \frac{e^{\kappa} - e^{-\kappa}}{2}$$

- BT implies rich underlying mathematical structure
- BT can be formulated as the canonical transformation of the Hamilton system

Construct Soliton Solutions: Hirota's method(I)

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travelling wave solution
(1-soliton solution)

$$q_n = \frac{1+e^{2\kappa(n-1)+2\beta t}}{1+e^{2\kappa n+2\beta t}}, \quad \beta = \sinh\kappa = \frac{e^{\kappa}-e^{-\kappa}}{2}$$

$$e^{q_n} = \frac{\tau_{n-1}}{\tau_n} \quad \text{or} \quad q_n = \log\frac{\tau_{n-1}}{\tau_n}$$

$$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \rightarrow \frac{d^2}{dt^2}\log\tau_{n-1} - \frac{d^2}{dt^2}\log\tau_n = \frac{\tau_{n-2}\tau_n}{\tau_{n-1}^2} - \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}$$

$$\rightarrow \frac{d^2}{dt^2}\log\tau_{n-1} - \frac{\tau_{n-2}\tau_n}{\tau_{n-1}^2} = \frac{d^2}{dt^2}\log\tau_n - \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} = f(t) \rightarrow \boxed{\tau_n''\tau_n - \tau_n'^2 = \tau_{n-1}\tau_{n+1} - f(t)\,\tau_n^2} \quad (**)$$

- **Hirota's bilinear differential operator (D-operator)**

$$D_x^m D_t^n f\cdot g = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n f(x,t)g(x',t')\Big|_{x=x',t=t'}$$

$$D_x f\cdot g = f_x g - f g_x, \quad D_x^2 f\cdot g = f_{xx}g - 2f_x g_x + f g_{xx}, \quad D_x D_t f\cdot g = f_{xt}g - f_x g_t - f_t g_x + f g_{xt}, \quad \text{etc.}$$

(**) $\longrightarrow$

$$\frac{1}{2}D_t^2\,\tau_n\cdot\tau_n = \tau_{n+1}\tau_{n-1} - f(t)\,\tau_n^2$$

"Bilinear equation (form)" of Toda lattice

τ_n : **tau function**

Example of Hirota derivative

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Hirota derivative (D-operator)

$$D_x^m D_t^n \, f \cdot g = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n \, f(x,t) g(x',t') \Big|_{x=x',t=t'}$$

$$D_x \, f \cdot g = (\partial_x - \partial_{x'}) \, f(x) g(x') \Big|_{x=x'} = f_x(x) g(x') - f(x) g_{x'}(x') \Big|_{x=x'} = f' g - f g'$$

$$D_t^2 \, f \cdot g = (\partial_t - \partial_{t'})^2 \, f(t) g(t') \Big|_{t=t'} = \left(\partial_t^2 - 2\partial_t \partial_{t'} + \partial_{t'}^2\right) f(t) g(t') \Big|_{t=t'}$$

$$= \quad f_{tt}(t) g(t') - 2 f_t(t) g_{t'}(t') + f(t) g_{t't'}(t') \Big|_{t=t'} = f'' g - 2 f' g' + f g''$$

問： Calculate $D_x D_y$ f・g from the definition.

$$D_x D_y \, f \cdot g = (\partial_x - \partial_{x'}) \left(\partial_y - \partial_{y'}\right) f(t) g(t') \Big|_{x=x',y=y'} = \left(\partial_x \partial_y - \partial_x \partial_{y'} - \partial_y \partial_{x'} + \partial_{x'} \partial_{y'}\right) f(x,y) g(x',y') \Big|_{x=x,y=y'}$$

$$= \quad f_{xy}(x,y) g(x',y') - f_x(x,y) g_{y'}(x',y') - f_y(x,y) g_{x'}(x',y') + f(x,y) g_{x'y'}(x',y') \Big|_{x=x',y=y'}$$

$$= \quad f_{xy} g - f_x g_y - f_y g_x + f g_{xy}$$

Leibnitz rule + signs

Construction of Soliton Solutions: Hirota's Method:

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$$\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \longrightarrow q_n = \log \frac{\tau_{n-1}}{\tau_n} \longrightarrow \frac{1}{2} D_t^2\, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - f(t)\, \tau_n^2$$

$$D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^m\, (\partial_t - \partial_{t'})^n\, f(x,t) g(x',t')\Big|_{x=x',t=t'}$$

Properties of D-operator:

Bilinearity:	$D_x^m D_t^n\, (af + bg)\cdot h = aD_x^m D_t^n\, f\cdot h + bD_x^m D_t^n\, g \cdot h$
Exchange rule:	$D_x^m D_t^n\, f \cdot g = (-1)^{m+n} D_x^m D_t^n\, g \cdot f$
constant argument:	$D_x^m D_t^n\, f \cdot 1 = \partial_x^m \partial_t^n\, f$
Rule for exponential fns.	$D_x^m D_t^n\, e^{p_1 x + q_1 t} \cdot e^{p_2 x + q_2 t} = (p_1 - p_2)^m (q_1 - q_2)^n\, e^{(p_1+p_2)x + (q_1+q_2)t}$

Construction of soliton solutions

- $q_n = 0$ is a solution. Correspondingly, $\tau_n = 1$ is a solution ($f(t)=1$).
- Apply perturbational technique to $\tau_n = 1$. Namely, assume the expansion

$$\tau_n = 1 + \epsilon f_n^{(1)} + \epsilon^2 f_n^{(2)} + \epsilon^3 f_n^{(3)} + \cdots$$

and plug it in the bilinear equation. Solve the equations obtained from coefficients of ϵ^j from the lower order. Stop this process at appropriate order and we have an approximate solution.

Construction of Soliton Solutions: Hirota's Method:

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$$\frac{1}{2}D_t^2\,\tau_n\cdot\tau_n=\tau_{n+1}\tau_{n-1}-\tau_n^2,\qquad \tau_n=1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}+\cdots$$

$$\frac{1}{2}D_t^2\,\left(1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}\right)\cdot\left(1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}\right)$$

$$=\left(1+\epsilon f_{n+1}^{(1)}+\epsilon^2 f_{n+1}^{(2)}+\epsilon^3 f_{n+1}^{(3)}\right)\left(1+\epsilon f_{n-1}^{(1)}+\epsilon^2 f_{n-1}^{(2)}+\epsilon^3 f_{n-1}^{(3)}\right)-\left(1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}\right)^2$$

$$O(\epsilon):\qquad f_n^{(1)\prime\prime}=f_{n+1}^{(1)}+f_{n-1}^{(1)}-2f_n^{(1)}$$

$$O(\epsilon^2):\quad f_n^{(2)\prime\prime}-f_{n+1}^{(2)}-f_{n-1}^{(2)}+2f_n^{(2)}=-\frac{1}{2}D_t^2\,f_n^{(1)}\cdot f_n^{(1)}+f_{n+1}^{(1)}f_{n-1}^{(1)}-f_n^{(1)2}$$

$$O(\epsilon^3):\qquad f_n^{(3)\prime\prime}-f_{n+1}^{(3)}-f_{n-1}^{(3)}+2f_n^{(3)}=-D_t^2\,f_n^{(1)}\cdot f_n^{(2)}+f_{n+1}^{(1)}f_{n-1}^{(2)}+f_{n+1}^{(2)}f_{n-1}^{(1)}-2f_n^{(1)}f_n^{(2)}$$

- **1-soliton solution:** $f_n^{(1)}=e^{\eta_1},\quad \eta_1=P_1 n+Q_1 t\ (+\text{const.})$

$$O(\epsilon):\quad Q_1^2=e^{P_1}+e^{-P_1}-2=(e^{\frac{P_1}{2}}-e^{-\frac{P_1}{2}})^2\ \rightarrow\ Q_1=\pm 2\sinh\frac{P_1}{2}$$

$$O(\epsilon^2):\quad f_n^{(2)\prime\prime}-f_{n+1}^{(2)}-f_{n-1}^{(2)}+2f_n^{(2)}=-\frac{1}{2}D_t^2\,f_n^{(1)}\cdot f_n^{(1)}+f_{n+1}^{(1)}f_{n-1}^{(1)}-f_n^{(1)2}=0$$

⟶ We can choose $f_n^{(2)}=0.$ Similarly, we have $f_n^{(k)}=0\quad (k=3,4,\ldots.)$

⟶ Perturbation is truncated! We have an EXACT solution!

1-soliton solution: $\tau_n = 1 + e^{\eta_1}, \quad \eta_1 = P_1 n \pm 2\sinh\frac{P_1}{2} t$

- **2-soliton solution:** $f_n^{(1)} = e^{\eta_1} + e^{\eta_2}, \quad \eta_i = P_i n + Q_i t \ (+\text{const.})$

$$O(\epsilon): \qquad f_n^{(1)\prime\prime} = f_{n+1}^{(1)} + f_{n-1}^{(1)} - 2f_n^{(1)} \longrightarrow \qquad Q_i = \pm 2\sinh\frac{P_i}{2}$$

$$O(\epsilon^2): \quad f_n^{(2)\prime\prime} - f_{n+1}^{(2)} - f_{n-1}^{(2)} + 2f_n^{(2)} = -\frac{1}{2}D_t^2\, f_n^{(1)}\cdot f_n^{(1)} + f_{n+1}^{(1)} f_{n-1}^{(1)} - f_n^{(1)2}$$

$$= -\frac{1}{2}D_t^2\,(e^{\eta_1}+e^{\eta_2})\cdot(e^{\eta_1}+e^{\eta_2}) + \left(e^{\eta_1+P_1}+e^{\eta_2+P_2}\right)\left(e^{\eta_1-P_1}+e^{\eta_2-P_2}\right) - (e^{\eta_1}+e^{\eta_2})^2$$

$$= -D_t^2\, e^{\eta_1}\cdot e^{\eta_2} + e^{\eta_1+\eta_2+p_1-p_2} + e^{\eta_1+\eta_2-p_1+p_2} - 2e^{\eta_1+\eta_2}$$

$$= -(Q_1-Q_2)^2 e^{\eta_1+\eta_2} + \left(e^{\frac{P_1-P_2}{2}} - e^{-\frac{P_1-P_2}{2}}\right)^2 e^{\eta_1+\eta_2} = -\left(e^{\frac{P_1}{2}} - e^{-\frac{P_1}{2}}\right)\left(e^{\frac{P_2}{2}} - e^{-\frac{P_2}{2}}\right)\left(e^{\frac{P_1-P_2}{4}} - e^{-\frac{P_1-P_2}{4}}\right)^2 e^{\eta_1+\eta_2}$$

Put $f_n^{(2)} = A_{12} e^{\eta_1+\eta_2}$

$$\text{LHS} = A_{12}(Q_1+Q_2)^2 e^{\eta_1+\eta_2} - A_{12}\left(e^{\frac{P_1+P_2}{2}} - e^{-\frac{P_1+P_2}{2}}\right)^2 e^{\eta_1+\eta_2}$$

$$= -A_{12}\left(e^{\frac{P_1}{2}} - e^{-\frac{P_1}{2}}\right)\left(e^{\frac{P_2}{2}} - e^{-\frac{P_2}{2}}\right)\left(e^{\frac{P_1+P_2}{4}} - e^{-\frac{P_1+P_2}{4}}\right)^2 e^{\eta_1+\eta_2}$$

$$\longrightarrow \quad A_{12} = \left(\frac{e^{\frac{P_1-P_2}{4}} - e^{-\frac{P_1-P_2}{4}}}{e^{\frac{P_1+P_2}{4}} - e^{-\frac{P_1+P_2}{4}}}\right)^2 = \left(\frac{\sinh\frac{P_1-P_2}{4}}{\sinh\frac{P_1+P_2}{4}}\right)^2$$

Construction of Soliton Solutions: Hirota's Method:

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$$f_n^{(1)} = e^{\eta_1} + e^{\eta_2}, \quad f_n^{(2)} = A_{12} e^{\eta_1+\eta_2}, \quad \eta_i = P_i n + Q_i t, \quad Q_i = \pm 2 \sinh \frac{P_i}{2}$$

$$O(\epsilon^3): \quad f_n^{(3)\prime\prime} - f_{n+1}^{(3)} - f_{n-1}^{(3)} + 2 f_n^{(3)} = -D_t^2\, f_n^{(1)} \cdot f_n^{(2)} + f_{n+1}^{(1)} f_{n-1}^{(2)} + f_{n+1}^{(2)} f_{n-1}^{(1)} - 2 f_n^{(1)} f_n^{(2)}$$

$$= -D_t^2\, (e^{\eta_1} + e^{\eta_2}) \cdot A_{12} e^{\eta_1+\eta_2} + \left(e^{\eta_1+P_1} + e^{\eta_2+P_2}\right) A_{12} e^{\eta_1+\eta_2-P_1-P_2} + \left(e^{\eta_1-P_1} + e^{\eta_2-P_2}\right) A_{12} e^{\eta_1+\eta_2+P_1+P_2}$$
$$- 2\,(e^{\eta_1} + e^{\eta_2})\, A_{12} e^{\eta_1+\eta_2}$$

1st term of RHS:

$$D_t^2\, (e^{\eta_1} + e^{\eta_2}) \cdot e^{\eta_1+\eta_2} = D_t^2\, e^{\eta_1} \cdot e^{\eta_1+\eta_2} + D_t^2\, e^{\eta_2} \cdot e^{\eta_1+\eta_2} = [Q_1 - (Q_1 + Q_2)]^2\, e^{2\eta_1+\eta_2} + [Q_2 - (Q_1 + Q_2)]^2\, e^{\eta_1+2\eta_2}$$
$$= Q_2^2\, e^{2\eta_1+\eta_2} + Q_1^2\, e^{\eta_1+2\eta_2}$$

RHS: $$A_{12}\left[-Q_2^2\, e^{2\eta_1+\eta_2} - Q_1^2\, e^{\eta_1+2\eta_2} + \left(e^{\frac{P_2}{2}} - e^{-\frac{P_2}{2}}\right)^2 e^{2\eta_1+\eta_2} + \left(e^{\frac{P_1}{2}} - e^{-\frac{P_1}{2}}\right)^2 e^{\eta_1+2\eta_2}\right] = 0$$

→ We can choose $f_n^{(3)} = 0.$ Similarly, we have $f_n^{(k)} = 0 \quad (k = 4, 5, \ldots)$

→ Perturbation is truncated again! We have EXACT 2-soliton solution!

2-soliton solution: $\tau_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + e^{\eta_2}}, \quad A_{12} = \left(\dfrac{\sinh \frac{P_1-P_2}{4}}{\sinh \frac{P_1+P_2}{4}}\right)^2$

Why 2-soliton solution is “2-soliton”?

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2-soliton solution: $\tau_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2}, \quad \eta_i = p_i n + q_i t, \quad q_i = -2\sinh\frac{p_i}{2}$

Asuume $p_1 > p_2 > 0$ and write $\eta_i = p_i(n - v_i t), \quad v_i = \frac{2}{p_i}\sinh\frac{p_i}{2}$

Since v_i is monotnic increasing function of p_i, $v_1 > v_2 > 0$.

- **Look from the wave running with v_1 :** $\eta_1 = p_1(n - v_1 t) = p_1\xi_1 \quad \xi_1 =$ **const.**

Note: $\eta_2 = p_2(n - v_2 t) = p_2(n - v_1 t) + p_2(v_1 - v_2)t = p_2\xi_1 + p_2(v_1 - v_2)t \quad (v_1 - v_2 > 0)$

$$\tau_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} = 1 + e^{p_1\xi_1} + e^{p_2\xi_1 + p_2(v_1-v_2)t} + A_{12}e^{(p_1+p_2)\xi_1 + p_2(v_1-v_2)t}$$

equivalent up to trivial factor

$t \to -\infty \quad : \quad \tau_2 \sim 1 + e^{p_1\xi_1} = \boxed{1 + e^{\eta_1}}$

$t \to +\infty \quad : \quad \tau_2 \sim e^{p_2\xi_1 + p_2(v_1-v_2)t} + A_{12}e^{(p_1+p_2)\xi_1 + p_2(v_1-v_2)t} = e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} = e^{\eta_2}(1 + A_{12}e^{\eta_1}) \doteqdot \boxed{1 + A_{12}e^{\eta_1}}$

Remark : Mutiplying exp[Pt+Qn] to τ_n, $q_n = \log \tau_{n-1}/\tau_n$ changes only by constant.

at t~±∞ wave with velocity v_1 is observed, amplitude and velocity do not change. But phase shifted by log A_{12}.

Why 2-soliton solution is "2-soliton"?

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- **Look from the wave running with v_2 :** $\eta_2 = p_2(n - v_2 t) = p_2\xi_2 \quad \xi_2 =$ **const.**

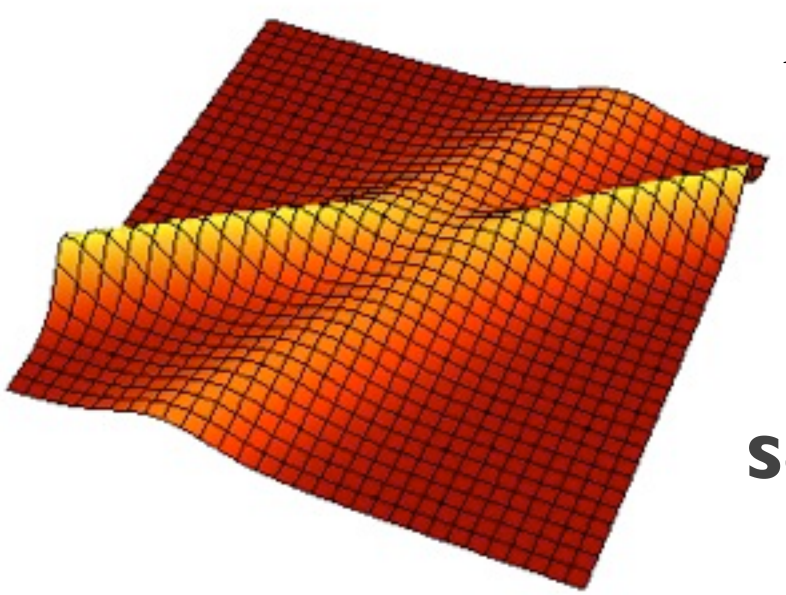
Note: $\eta_1 = p_1(n - v_1 t) = p_1(n - v_2 t) + p_1(v_2 - v_1)t = p_1\xi_2 + p_1(v_2 - v_1)t \quad (v_2 - v_1 < 0)$

$$\tau_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} = 1 + e^{p_1\xi_2 + p_1(v_2-v_1)t} + e^{p_2\xi_2} + A_{12}e^{(p_1+p_2)\xi_2 + p_1(v_2-v_1)t}$$

$$t \to -\infty \quad : \quad \tau_2 \sim e^{p_1\xi_2 + p_1(v_2-v_1)t} + A_{12}e^{(p_1+p_2)\xi_2 + p_1(v_2-v_1)t} = e^{\eta_1} + A_{12}e^{\eta_1+\eta_2} = e^{\eta_1}\left(1 + A_{12}e^{\eta_2}\right) \simeq \boxed{1 + A_{12}e^{\eta_2}}$$

$$t \to +\infty \quad : \quad \tau_2 \sim 1 + e^{p_2\xi_2} = \boxed{1 + e^{\eta_2}}$$

at t~±∞ wave with velocity v_2 is observed, amplitude and velocity do not change. But phase shifted by -log A_{12}.



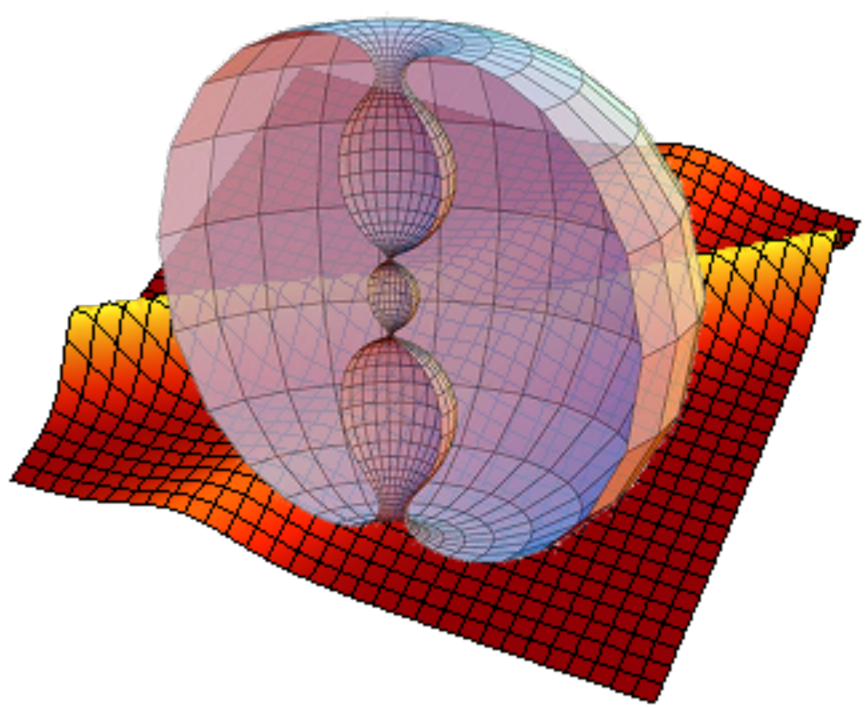
$$R_n = q_n - q_{n+1} = \log\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}$$

- **Amplitudes and velocities preserved .**
- **Phase shift as an evidence of nonlinear interaction**

Solitary wave with particle character: soliton = solitary+on

Chapter 2

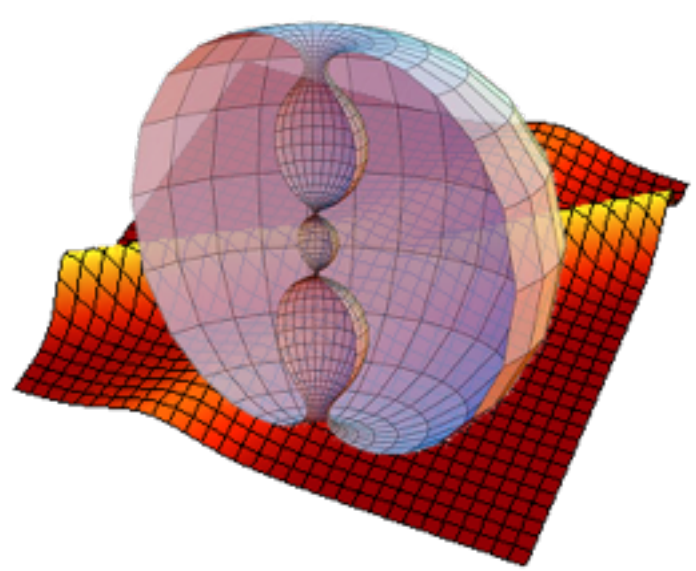
Theory of Integrable Systems through the two-dimensional Toda Lattice



I：2-d Toda lattice and its properties

Contents and Keywords

- Construct Solutions by Hirota's method
- Determinant Structure of Soliton Solutions：τ function
- Bilinear equation=identity of determinants：Plücker relation
- Molecule solution



Two-dimensional Toda Lattice

$$\frac{\partial^2 q_n}{\partial x \partial y} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}},$$

$$\frac{\partial^2 r_n}{\partial x \partial y} = e^{r_{n+1}} + e^{r_{n-1}} - 2e^{r_n},$$

$$\frac{\partial^2}{\partial x \partial y} \log(1+V_n) = V_{n+1} + V_{n-1} - 2V_n,$$

$$\frac{\partial}{\partial x} \log(1+V_n) = I_n - I_{n+1}, \qquad \frac{\partial I_n}{\partial y} = V_{n-1} - V_n$$

Bilinear equation： $\frac{1}{2} D_x D_y \, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - f(t)\, \tau_n^2$

Relations among dependent variables:

$$q_n = \log \frac{\tau_{n-1}}{\tau_n}, \quad r_n = q_n - q_{n+1} = \log \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad 1 + V_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad I_n = \frac{\partial q_n}{\partial x} = \frac{\partial}{\partial x} \log \frac{\tau_{n-1}}{\tau_n}$$

Relation to Toda lattice: t = x+y, s = x-y and impose $\frac{\partial q_n}{\partial s} = 0$

$$\frac{\partial^2 q_n}{\partial x \partial y} = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \;\to\; \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) q_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \;\to\; \frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}}$$

Soliton Solution

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$$\frac{1}{2}D_xD_y\,\tau_n\cdot\tau_n=\tau_{n+1}\tau_{n-1}-\tau_n^2,\qquad \tau_n=1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}+\cdots$$

$O(\epsilon)$:
$$\partial_x\partial_y f_n^{(1)}=f_{n+1}^{(1)}+f_{n-1}^{(1)}-2f_n^{(1)}$$

$O(\epsilon^2)$:
$$\partial_x\partial_y f_n^{(2)}-f_{n+1}^{(2)}-f_{n-1}^{(2)}+2f_n^{(2)}=-\frac{1}{2}D_xD_y\,f_n^{(1)}\cdot f_n^{(1)}+f_{n+1}^{(1)}f_{n-1}^{(1)}-f_n^{(1)2}$$

$O(\epsilon^3)$:
$$\partial_x\partial_y f_n^{(3)}-f_{n+1}^{(3)}-f_{n-1}^{(3)}+2f_n^{(3)}=-D_xD_y\,f_n^{(1)}\cdot f_n^{(2)}+f_{n+1}^{(1)}f_{n-1}^{(2)}+f_{n+1}^{(2)}f_{n-1}^{(1)}-2f_n^{(1)}f_n^{(2)}$$

1-soliton solution： $f_n^{(1)}=R_1^{2n}e^{P_1x+Q_1y}=e^{\zeta_1},\quad \zeta_1=2n\log R_1+P_1x+Q_1y\ (+\zeta_{10})$

$O(\epsilon)$:
$$P_1Q_1=R_1^2+\frac{1}{R_1^2}-2=\left(R_1-\frac{1}{R_1}\right)^2$$

$O(\epsilon^2)$:
$$f_n^{(2)\prime\prime}-f_{n+1}^{(2)}-f_{n-1}^{(2)}+2f_n^{(2)}=-\frac{1}{2}D_xD_y\,f_n^{(1)}\cdot f_n^{(1)}+f_{n+1}^{(1)}f_{n-1}^{(1)}-f_n^{(1)2}=0\ \rightarrow\ \boxed{f_n^{(2)}=0}$$

1-soliton solution： $\tau_n=1+R_1^{2n}e^{P_1x+Q_1y},\quad P_1Q_1=\left(R_1-\frac{1}{R_1}\right)^2$

Soliton Solution (2)

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$$\frac{1}{2}D_xD_y\,\tau_n\cdot\tau_n=\tau_{n+1}\tau_{n-1}-\tau_n^2,\quad \tau_n=1+\epsilon f_n^{(1)}+\epsilon^2 f_n^{(2)}+\epsilon^3 f_n^{(3)}+\cdots$$

$$O(\epsilon):\qquad \partial_x\partial_y f_n^{(1)}=f_{n+1}^{(1)}+f_{n-1}^{(1)}-2f_n^{(1)}$$

$$O(\epsilon^2):\qquad \partial_x\partial_y f_n^{(2)}-f_{n+1}^{(2)}-f_{n-1}^{(2)}+2f_n^{(2)}=-\frac{1}{2}D_xD_y\,f_n^{(1)}\cdot f_n^{(1)}+f_{n+1}^{(1)}f_{n-1}^{(1)}-f_n^{(1)2}$$

$$O(\epsilon^3):\qquad \partial_x\partial_y f_n^{(3)}-f_{n+1}^{(3)}-f_{n-1}^{(3)}+2f_n^{(3)}=-D_xD_y\,f_n^{(1)}\cdot f_n^{(2)}+f_{n+1}^{(1)}f_{n-1}^{(2)}+f_{n+1}^{(2)}f_{n-1}^{(1)}-2f_n^{(1)}f_n^{(2)}$$

1-soliton solution： $\tau_n=1+R_1^{2n}e^{P_1x+Q_1y},\quad P_1Q_1=\left(R_1-\frac{1}{R_1}\right)^2$

問：Similar to Toda lattice, construct 2-soliton solution to two-dimensional Toda lattice.

Hint: **Put** $f_n^{(1)}=e^{\zeta_1}+e^{\zeta_2},\quad \zeta_i=2n\log R_i+P_ix+Q_iy\ (+\zeta_{i0})$

2-Soliton Solution (I)

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$$f_n^{(1)} = e^{\zeta_1} + e^{\zeta_2}, \quad \zeta_i = 2n\log R_i + P_i x + Q_i y$$

$$O(\epsilon): \quad \partial_x\partial_y f_n^{(1)} = f_{n+1}^{(1)} + f_{n-1}^{(1)} - 2f_n^{(1)} \quad \Longrightarrow P_i Q_i = \left(R_i - \frac{1}{R_i}\right)^2$$

$$O(\epsilon^2): \quad \partial_x\partial_y f_n^{(2)} - f_{n+1}^{(2)} - f_{n-1}^{(2)} + 2f_n^{(2)} = -\frac{1}{2}D_x D_y\, f_n^{(1)}\cdot f_n^{(1)} + f_{n+1}^{(1)} f_{n-1}^{(1)} - f_n^{(1)2}$$

$$= -\frac{1}{2}D_x D_y\,(e^{\zeta_1}+e^{\zeta_2})\cdot(e^{\zeta_1}+e^{\zeta_2}) + \left(R_1^2 e^{\zeta_1} + R_2^2 e^{\zeta_2}\right)\left(R_1^{-2}e^{\zeta_1} + R_2^{-2}e^{\zeta_2}\right)$$
$$-\left(e^{\zeta_1}+e^{\zeta_2}\right)^2$$

$$= -\left[(P_1-P_2)(Q_1-Q_2) - \left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right)^2\right] e^{\zeta_1+\zeta_2}$$

Put $f_n^{(2)} = A_{12} e^{\zeta_1+\zeta_2}$

$$\text{左辺} = \left[(P_1+P_2)(Q_1+Q_2) - \left(R_1R_2 - \frac{1}{R_1R_2}\right)^2\right] A_{12} e^{\zeta_1+\zeta_2}$$

$$\Longrightarrow \quad f_n^{(2)} = A_{12} e^{\zeta_1+\zeta_2}, \quad A_{12} = -\frac{\left[(P_1-P_2)(Q_1-Q_2) - \left(\frac{R_1}{R_2}-\frac{R_2}{R_1}\right)^2\right]}{\left[(P_1+P_2)(Q_1+Q_2) - \left(R_1R_2 - \frac{1}{R_1R_2}\right)^2\right]}$$

2-Soliton Solution (2)

$$O(\epsilon^3): \quad \partial_x\partial_y f_n^{(3)} - f_{n+1}^{(3)} - f_{n-1}^{(3)} + 2f_n^{(3)} = -D_xD_y\, f_n^{(1)}\cdot f_n^{(2)} + f_{n+1}^{(1)}f_{n-1}^{(2)} + f_{n+1}^{(2)}f_{n-1}^{(1)} - 2f_n^{(1)}f_n^{(2)}$$

$$= -\boxed{D_xD_y\left(e^{\zeta_1}+e^{\zeta_2}\right)\cdot A_{12}e^{\zeta_1+\zeta_2}} + \left(R_1^2e^{\zeta_1}+R_2^2e^{\zeta_2}\right)A_{12}\frac{1}{R_1^2R_2^2}e^{\zeta_1+\zeta_2} + \left(\frac{1}{R_1^2}e^{\zeta_1}+\frac{1}{R_2^2}e^{\zeta_2}\right)A_{12}R_1^2R_2^2e^{\zeta_1+\zeta_2}$$

$$-2\left(e^{\zeta_1}+e^{\zeta_2}\right)A_{12}e^{\zeta_1+\zeta_2}$$

右辺第 1 項： $D_xD_y\left(e^{\zeta_1}+e^{\zeta_2}\right)\cdot e^{\zeta_1+\zeta_2} = D_xD_y\, e^{\zeta_1}\cdot e^{\zeta_1+\zeta_2} + D_xD_y\, e^{\zeta_2}\cdot e^{\zeta_1+\zeta_2}$

$$= [P_1-(P_1+P_2)][Q_1-(Q_1+Q_2)]\,e^{2\zeta_1+\zeta_2}\,[P_2-(P_1+P_2)][Q_2-(Q_1+Q_2)]\,e^{\zeta_1+2\zeta_2}$$

$$= P_2Q_2e^{2\zeta_1+\zeta_2} + P_1Q_1e^{\zeta_1+2\zeta_2}$$

右辺 $= A_{12}\left[\left\{-P_2Q_2+\frac{1}{R_2^2}+R_2^2-2\right\}e^{2\zeta_1+\zeta_2} + \left\{-P_1Q_1+\frac{1}{R_1^2}+R_1^2-2\right\}e^{\zeta_1+2\zeta_2}\right] = 0 \quad \because\ P_iQ_i = \left(R_i-\frac{1}{R_i}\right)^2$

$$O(\epsilon^3): \quad \partial_x\partial_y f_n^{(3)} - f_{n+1}^{(3)} - f_{n-1}^{(3)} + 2f_n^{(3)} = 0 \quad \Longrightarrow \quad \boxed{f_n^{(3)} = 0 \text{ と取れる！}}$$

Similary, one can choose $f_n^{(4)} = f_n^{(5)} = \ldots = 0$. Finally we have

2-soliton solution:

$$\tau_n = 1 + e^{\zeta_1} + e^{\zeta_2} + A_{12}e^{\zeta_1+\zeta_2}, \quad A_{12} = -\frac{\left[(P_1-P_2)(Q_1-Q_2) - \left(\frac{R_1}{R_2}-\frac{R_2}{R_1}\right)^2\right]}{\left[(P_1+P_2)(Q_1+Q_2) - \left(R_1R_2-\frac{1}{R_1R_2}\right)^2\right]}$$

Remarks

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- **3-soliton solution:** $\tau_n = 1 + e^{\zeta_1} + e^{\zeta_2} + e^{\zeta_3} + A_{12}e^{\zeta_1+\zeta_2} + A_{23}e^{\zeta_2+\zeta_3} + A_{13}e^{\zeta_1+\zeta_3} + A_{123}e^{\zeta_1+\zeta_2+\zeta_3},$

$$\zeta_i = 2n\log R_i + P_i x + Q_i y + \zeta_{i0}, \quad P_iQ_i = \left(R_i - \frac{1}{R_i}\right)^2,$$

$$A_{ij} = -\frac{\left[(P_i - P_j)(Q_i - Q_j) - \left(\frac{R_i}{R_j} - \frac{R_j}{R_i}\right)^2\right]}{\left[(P_i + P_j)(Q_i + Q_j) - \left(R_iR_j - \frac{1}{R_iR_j}\right)^2\right]}, \quad A_{123} = A_{12}A_{23}A_{13}$$

- **Clever parametrization of soliton solutions (important!)**

$$\boxed{P_i = p_i - q_i, \quad Q_i = -\frac{1}{p_i} + \frac{1}{q_i}, \quad R_i = \left(\frac{p_i}{q_i}\right)^{\frac{1}{2}}} \implies$$

$$P_iQ_i = (p_i - q_i)\left(-\frac{1}{p_i} + \frac{1}{q_i}\right) = \frac{p_i}{q_i} + \frac{q_i}{p_i} - 2 = \left(R_i - \frac{1}{R_i}\right)^2$$

$$A_{ij} = \frac{(p_i - q_i)(p_j - q_j)}{(p_i - q_j)(p_j - q_i)}$$

2-soliton solution: $\tau_n = 1 + e^{\eta_1-\xi_1} + e^{\eta_2-\xi_2} + A_{12}e^{\eta_1+\eta_2-\xi_1-\xi_2}$

$$\eta_i = n\log p_i + p_i x - \frac{y}{p_i} + \eta_{0i}, \quad \xi_i = n\log q_i + q_i x - \frac{y}{q_i} + \xi_{0i}, \quad A_{12} = \frac{(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}$$

Casorati Determinant (I)

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2-soliton solution is expessed by 2x2 determinant!

$$\tau_n = 1 + e^{\eta_1-\xi_1} + e^{\eta_2-\xi_2} + A_{12}e^{\eta_1+\eta_2-\xi_1-\xi_2} \simeq \begin{vmatrix} e^{\eta_1}+e^{\xi_1} & p_1e^{\eta_1}+q_1e^{\xi_1} \\ e^{\eta_2}+e^{\xi_2} & p_2e^{\eta_2}+q_2e^{\xi_2} \end{vmatrix} = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} \end{vmatrix}$$

Casorati determinant

$$f_n^{(i)} = e^{\eta_i} + e^{\xi_i}, \quad \eta_i = n\log p_i + p_i x - \frac{y}{p_i} + \eta_{0i}, \quad \xi_i = n\log q_i + q_i x - \frac{y}{q_i} + \xi_{0i}$$

Check

$$右辺 = \left(e^{\eta_1}+e^{\xi_1}\right)\left(p_2e^{\eta_2}+q_2e^{\xi_2}\right) - \left(e^{\eta_2}+e^{\xi_2}\right)\left(p_1e^{\eta_1}+q_1e^{\xi_1}\right)$$

$$= (p_2-p_1)e^{\eta_1+\eta_2} + (q_2-p_1)e^{\eta_1+\xi_2} + (p_2-q_1)e^{\eta_2+\xi_1} + (q_2-q_1)e^{\xi_1+\xi_2}$$

$$\simeq 1 + \frac{q_2-p_1}{q_2-q_1}e^{\eta_1-\xi_1} + \frac{p_2-q_1}{q_2-q_1}e^{\eta_2-\xi_2} + \frac{p_2-p_1}{q_2-q_1}e^{\eta_1+\eta_2-\xi_1-\xi_2} \qquad (\bigstar)$$

Use the freedom of arbitrary constant:

$$(q_2-p_1)e^{\eta_1} = e^{\eta_1+\log(q_2-p_1)} \to e^{\eta_1} \qquad (p_2-q_1)e^{\eta_2} = e^{\eta_2+\log(p_2-q_1)} \to e^{\eta_2}$$

$$(q_2-q_1)e^{\xi_1} = e^{\xi_1+\log(q_2-q_1)} \to e^{\xi_1} \qquad (q_2-q_1)e^{\xi_2} = e^{\xi_2+\log(q_2-q_1)} \to e^{\xi_2}$$

$$(\bigstar) = 1 + e^{\eta_1-\xi_1} + e^{\eta_2-\xi_2} + \frac{p_2-p_1}{q_2-q_1} \times \frac{(q_2-q_1)^2}{(q_2-p_1)(p_2-q_1)}e^{\eta_1+\eta_2-\xi_1-\xi_2}$$

$$= 1 + e^{\eta_1-\xi_1} + e^{\eta_2-\xi_2} + \frac{(p_2-p_1)(q_2-q_1)}{(q_2-p_1)(p_2-q_1)}e^{\eta_1+\eta_2-\xi_1-\xi_2} = (左辺)$$

Casorati Determinant (2)

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Theorem： The following N x N Casorati determinant

$$\tau_n = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}$$

satisfies the bilinear equation of two-dimensional Toda lattice

$$\frac{1}{2} D_x D_y \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

where, $f_n^{(k)}$ (k=1,...,N) satisfies the following linear relations:

$$\frac{\partial f_n^{(k)}}{\partial x} = f_{n+1}^{(k)}, \qquad \frac{\partial f_n^{(k)}}{\partial y} = -f_{n-1}^{(k)}$$

Remark: $f_n^{(k)} = p_k^n \exp\left(p_k x - \frac{y}{p_k} + \eta_{k0}\right) + q_k^n \exp\left(q_k x - \frac{y}{q_k} + \xi_{k0}\right)$ ⟶ N-soliton solution

Casorati Determinant (3)

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- **Step 1: derivative of τ = determinant with shifted columns: "Diffrential formula"**

Freeman-Nimmo's notation:

$$\tau_n = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix} = |\, \mathbf{0}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N-1} \,|, \quad \boldsymbol{j} = \begin{pmatrix} f_{n+j}^{(1)} \\ f_{n+j}^{(2)} \\ \vdots \\ f_{n+j}^{(N)} \end{pmatrix}$$

Prop: (Differential Formula)

$$\tau_n = |\, \mathbf{0}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N-1} \,| \qquad \partial_x \tau_n = |\, \mathbf{0}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N} \,|$$

$$\tau_{n+1} = |\, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N-1}, \boldsymbol{N} \,| \qquad -\partial_y \tau_n = |\, \mathbf{-1}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N-1} \,|$$

$$\tau_{n-1} = |\, \mathbf{-1}, \mathbf{0}, \mathbf{1}, \cdots, \boldsymbol{N-2} \,| \qquad -\left(\partial_x \partial_y + 1\right) \tau_n = |\, \mathbf{-1}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N} \,|$$

Casorati Determinant (4)

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- **Check: Formulas in the left are trivial. Right ones are checked as:**

$$\partial_x \tau_n = \left| \mathbf{0}', \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \right| + \cdots + \left| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}', \mathbf{N-1} \right| + \left| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}' \right|$$

$$= | \mathbf{1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} | + \cdots + | \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}, \mathbf{N-1} | + | \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N} |$$

$$= \boxed{| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N} |} \qquad \because \frac{\partial f_n^{(i)}}{\partial x} = f_{n+1}^{(i)}$$

$$\partial_y \tau_n = \left| \mathbf{0}', \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \right| + \left| \mathbf{0}, \mathbf{1}', \cdots, \mathbf{N-2}, \mathbf{N-1} \right| + \cdots + \left| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}' \right|$$

$$= -| \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} | - | \mathbf{0}, \mathbf{0}, \cdots, \mathbf{N-2}, \mathbf{N-1} | - \cdots - | \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-2} |$$

$$= \boxed{-| \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} |} \qquad \because \frac{\partial f_n^{(i)}}{\partial y} = -f_{n-1}^{(i)}$$

$$\partial_x \partial_y \tau_n = -\left| \mathbf{-1}', \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} \right| - \left| \mathbf{-1}, \mathbf{1}', \cdots, \mathbf{N-2}, \mathbf{N-1} \right| + \cdots + \left| \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1}' \right|$$

$$= -| \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N-1} | - | \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N} |$$

$$= \boxed{-\tau_n - | \mathbf{-1}, \mathbf{1}, \cdots, \mathbf{N-2}, \mathbf{N} |}$$

Casorati Determinant (5)

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- **Step 2: Bilinear eq. = identity of determinant: Plücker relation**

$$0=\frac{1}{2}D_xD_y\tau_n\cdot\tau_n-\tau_{n+1}\tau_{n-1}+\tau_n^2=\left(\partial_x\partial_y\tau_n\right)\tau_n-(\partial_x\tau_n)\left(\partial_y\tau_n\right)-\tau_{n+1}\tau_{n-1}+\tau_n^2$$

$$=\Big(-|\mathbf{0,1,\cdots,N-2,N-1}|-|\mathbf{-1,1,\cdots,N-2,N}|\Big)\times|\mathbf{0,1,\cdots,N-2,N-1}|$$
$$-|\mathbf{0,1,\cdots,N-2,N}|\times\Big(-|\mathbf{-1,1,\cdots,N-2,N-1}|\Big)$$
$$-|\mathbf{1,2,\cdots,N-1,N}|\times|\mathbf{-1,0,1,\cdots,N-2}|$$
$$+|\mathbf{0,1,\cdots,N-2,N-1}|\times|\mathbf{0,1,\cdots,N-2,N-1}|$$

$$=-|\mathbf{-1,0,1,\cdots,N-2}|\times|\mathbf{1,2,\cdots,N-1,N}|$$
$$+|\mathbf{-1,1,\cdots,N-2,N-1}|\times|\mathbf{0,1,\cdots,N-2,N}|$$
$$-|\mathbf{-1,1,\cdots,N-2,N}|\times|\mathbf{0,1,\cdots,N-2,N-1}|$$

Bilinear eq. of 2DTL

$$(※)\quad 0=|\mathbf{-1,0,1,\cdots,N-2}|\times|\mathbf{1,\cdots,N-2,N-1,N}|$$
$$+|\mathbf{0,1,\cdots,N-2,N-1}|\times|\mathbf{-1,1,\cdots,N-2,N}|$$
$$-|\mathbf{0,1,\cdots,N-2,N}|\times|\mathbf{-1,1,\cdots,N-2,N-1}|$$

Casorati Determinant (6)

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- **Proposition: Laplace expansion of determinant**

$A = (a_{ij})_{1\le i,j\le N}$: N x N matrix

$|A|^{i_1 i_2\cdots i_l}_{j_1 j_2\cdots j_l}$: ℓ x ℓ minor determinant obtained by choosing $i_l, i_2 \ldots, i_\ell$-th rows and $j_l, j_2, \ldots, j_\ell$-th columns from A

$\overline{|A|}^{i_1 i_2\cdots i_l}_{j_1 j_2\cdots j_l}$: (N-ℓ)x (N-ℓ) minor determinant obtained by removing $i_l, i_2 \ldots, i_\ell$-th rows and $j_l, j_2, \ldots, j_\ell$-th columns from A

Fix ℓ integers $i_l, i_2 \ldots, i_\ell$ such that $1 \le i_1 < i_2 < \cdots < i_l \le N$ Then we have:

$$|A| = (-1)^{i_1+\cdots+i_l} \sum_{1\le j_1<\cdots<j_l\le N} (-1)^{j_1+\cdots+j_l} \quad |A|^{i_1 i_2\cdots i_l}_{j_1 j_2\cdots j_l} \times \overline{|A|}^{i_1 i_2\cdots i_l}_{j_1 j_2\cdots j_l}$$

- **Example:** $\ell = 1, i_l = 1.$ $\quad |A|^{i_1}_{j_1} = a_{1 j_1}$

$$|A| = \sum_{1\le j_1\le N} (-1)^{1+j_1} a_{1 j_1} \times \overline{|A|}^{1}_{j_1} = \sum_{1\le j_1\le N} a_{1 j_1} \times A_{1 j_1}, \quad A_{1 j_1} :$$ (1,j_l)-cofactor

→ Expansion w.r.t. 1st row

Casorati Determinant (7)

Consider the following identity of 2N x 2N determinant:

$$0 = \left| \begin{array}{c:cccc:ccc:cc} \mathbf{-1} & \mathbf{0} & \mathbf{1} & \cdots & \boldsymbol{N-2} & & \emptyset & & \boldsymbol{N-1} & \boldsymbol{N} \\ \hdashline \mathbf{-1} & & \emptyset & & & \mathbf{1} & \cdots & \boldsymbol{N-2} & \boldsymbol{N-1} & \boldsymbol{N} \end{array} \right|$$

$$\underbrace{\qquad\qquad\qquad}_{N-1} \quad \underbrace{\qquad\qquad}_{N-2}$$

- Verification: Subtract the lower block from the upper block, then

$$\text{RHS} = \left| \begin{array}{c:c:ccc:ccc:cc} \emptyset & \mathbf{0} & \mathbf{1} & \cdots & \boldsymbol{N-2} & -(\mathbf{1}) & \cdots & -(\boldsymbol{N-2}) & \emptyset & \emptyset \\ \hdashline \mathbf{-1} & & & \emptyset & & \mathbf{1} & \cdots & \boldsymbol{N-2} & \boldsymbol{N-1} & \boldsymbol{N} \end{array} \right|$$

$$= \left| \begin{array}{c:cccc:ccc:cc} \emptyset & \mathbf{0} & \mathbf{1} & \cdots & \boldsymbol{N-2} & & \emptyset & & \emptyset & \emptyset \\ \hdashline \mathbf{-1} & & \emptyset & & & \mathbf{1} & \cdots & \boldsymbol{N-2} & \boldsymbol{N-1} & \boldsymbol{N} \end{array} \right|$$

$$\underbrace{\qquad\qquad\qquad}_{N-1} \quad \underbrace{\qquad\qquad}_{N-2}$$

Now apply the Laplace expansion with ℓ=N, i_l=1,...,i_N=N. Since the upper block contains N+1 empty columns, all the terms in the expansion are 0.

Casorati Determinant (8)

$$0 = \left| \begin{array}{c|cccc|ccc|cc} -1 & 0 & 1 & \cdots & N-2 & & \emptyset & & N-1 & N \\ \hline -1 & & \emptyset & & & 1 & \cdots & N-2 & N-1 & N \end{array} \right|$$

Apply the Laplace expansion to RHS directly: Plücker relation (simplest case)

$$\begin{aligned} 0 = \; & |-1, 0, 1, \cdots, N-2| \times |1, \cdots, N-2, N-1, N| \\ + \; & |0, 1, \cdots, N-2, N-1| \times |-1, 1, \cdots, N-2, N| \\ - \; & |0, 1, \cdots, N-2, N| \times |-1, 1, \cdots, N-2, N-1| \end{aligned}$$

→ Bilinear equation

Bilinear equation of 2DTL = Plücker relation

$$\tau_n = \begin{vmatrix} \varphi_n^{(1)} & \varphi_{n+1}^{(1)} & \cdots & \varphi_{n+N-1}^{(1)} \\ \varphi_n^{(2)} & \varphi_{n+1}^{(2)} & \cdots & \varphi_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_n^{(N)} & \varphi_{n+1}^{(N)} & \cdots & \varphi_{n+N-1}^{(N)} \end{vmatrix} \longrightarrow \frac{1}{2} D_x D_y \, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

$$\frac{\partial \varphi_n^{(i)}}{\partial x} = \varphi_{n+1}^{(i)}, \qquad \frac{\partial \varphi_n^{(i)}}{\partial y} = -\varphi_{n-1}^{(i)}$$

Essential Structure of Integrable Systems

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- **Infinite number of Plücker relations**
  - Distinguished column vectors -1, 0, N-1,N can be arbitrary
  - Number of distinguished column vectors is arbitrary (more than 4)

- **Differential/difference structure:**

  With appropriate differential/difference structure in τ, any determinant with arbitrary shift can be obtained by applying suitable differential operator to τ.

Example: introduce an infinite number of independent variables x_j, y_j ($j=1,2,\ldots$) such that

$$\frac{\partial \varphi_n^{(i)}}{\partial x_j} = \varphi_{n+j}^{(i)}, \qquad \frac{\partial \varphi_n^{(i)}}{\partial y_j} = -\varphi_{n-j}^{(i)}$$

- **Infinite number of Plücker relations**

  = Infinite number of bilinear equations sharing common solutions

  (with the above differential/difference structure) “2DTL hierarchy”

  (with x_j or y_j only) “KP hierarchy”

Sato Theory:

- Solution space of soliton equations is the universal Grassmann manifold
- τ functions are the Plücker coordinates.

Molecule solution (I)

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- **Casorati determinant solution**：　**n = phase of solitons**
- **“molecule solution”**：　**n = size of determinant**

Theorem：**n x n two-directional Wronski determinant**

$$\tau_n = \begin{vmatrix} f(x,y) & \partial_x f(x,y) & \cdots & \partial_x^{n-1} f(x,y) \\ \partial_y f(x,y) & \partial_x \partial_y f(x,y) & \cdots & \partial_x^{n-1}\partial_y f(x,y) \\ \vdots & \vdots & \cdots & \vdots \\ \partial_y^{n-1} f(x,y) & \partial_x \partial_y^{n-1} f(x,y) & \cdots & \partial_x^{n-1}\partial_y^{n-1} f(x,y) \end{vmatrix}$$

$f(x,y)$:　任意函数

satisfies the bilinear equation of 2DTL

$$\frac{1}{2} D_x D_y \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} \quad n = 0, 1, 2, \ldots,$$

初期 (境界) 条件：　$\tau_{-1} = 0, \quad \tau_0 = 1, \quad \tau_1 = f.$

- **Remark: the above boundary condition corresponds to:**

$$q_n = \log \frac{\tau_{n-1}}{\tau_n} \ : \ q_0 = -\infty, \quad r_n = \log \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} \ : \ r_0 = -\infty, \quad 1 + V_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} \ : \ 1 + V_0 = 0$$

Molecule solution (2)

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Proof： Reduce the bilinear equation to Plücker relation.

Key: identity among the determinants with different size.

$$\tau_{n+1} = \begin{vmatrix} f(x,y) & \cdots & \partial_x^{n-1} f(x,y) & \partial_x^n f(x,y) \\ \vdots & \cdots & \vdots & \vdots \\ \partial_y^{n-1} f(x,y) & \cdots & \partial_x^{n-1}\partial_y^{n-1} f(x,y) & \partial_x^n \partial_y^{n-1} f(x,y) \\ \partial_y^n f(x,y) & \cdots & \partial_x^{n-1}\partial_y^{n} f(x,y) & \partial_x^n \partial_y^n f(x,y) \end{vmatrix} = |\,\mathbf{0}, \cdots, \boldsymbol{n-1}, \boldsymbol{n}\,|, \quad \boldsymbol{j} = \begin{pmatrix} \partial_x^j f(x,y) \\ \vdots \\ \partial_x^j \partial_y^{n-1} f(x,y) \\ \partial_x^j \partial_y^n f(x,y) \end{pmatrix}$$

$$0 = \left|\begin{array}{cccc:ccc:c:cc} \mathbf{0} & \cdots & \boldsymbol{n-2} & \boldsymbol{n-1} & & \emptyset & & \boldsymbol{n} & \boldsymbol{\phi_1} & \boldsymbol{\phi_2} \\ \hdashline & & \emptyset & & \mathbf{0} & \cdots & \boldsymbol{n-2} & \boldsymbol{n} & \boldsymbol{\phi_1} & \boldsymbol{\phi_2} \end{array}\right|, \quad \phi_1 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Plücker Relation:

Key !

$$\begin{aligned} 0 = \; & |\,\mathbf{0}, \cdots, \boldsymbol{n-2}, \boldsymbol{n-1}, \boldsymbol{n}\,| \times |\,\mathbf{0}, \cdots, \boldsymbol{n-2}, \boldsymbol{\phi_1}, \boldsymbol{\phi_2}\,| \\ - \; & |\,\mathbf{0}, \cdots, \boldsymbol{n-2}, \boldsymbol{n-1}, \boldsymbol{\phi_1}\,| \times |\,\mathbf{0}, \cdots, \boldsymbol{n-2}, \boldsymbol{n}, \boldsymbol{\phi_2}\,| \\ + \; & |\,\mathbf{0}, \cdots, \boldsymbol{n-2}, \boldsymbol{n-1}, \boldsymbol{\phi_2}\,| \times |\,\mathbf{0}, \cdots, \boldsymbol{n-2}, \boldsymbol{n}, \boldsymbol{\phi_1}\,| \end{aligned}$$

Molecule solution (3)

Express each factor by τ：

$$
|\,0,\cdots,n-2,\phi_1,\phi_2\,| =
\begin{vmatrix}
f & \cdots & \partial_x^{n-2}f & 0 & 0 \\
\vdots & \cdots & \vdots & \vdots & \vdots \\
\partial_y^{n-2}f & \cdots & \partial_x^{n-2}\partial_y^{n-2}f & 0 & 0 \\
\partial_y^{n-1}f & \cdots & \partial_x^{n-2}\partial_y^{n-1}f & 1 & 0 \\
\partial_y^{n}f & \cdots & \partial_x^{n-2}\partial_y^{n}f & 0 & 1
\end{vmatrix}
=
\begin{vmatrix}
f & \cdots & \partial_x^{n-2}f \\
\vdots & \cdots & \vdots \\
\partial_y^{n-2}f & \cdots & \partial_x^{n-2}\partial_y^{n-2}f
\end{vmatrix}
= \tau_{n-1}
$$

$$
|\,0,\cdots,n-2,n-1,\phi_1\,| =
\begin{vmatrix}
f & \cdots & \partial_x^{n-2}f & \partial_x^{n-1} & 0 \\
\vdots & \cdots & \vdots & \vdots & \vdots \\
\partial_y^{n-2}f & \cdots & \partial_x^{n-2}\partial_y^{n-2}f & \partial_x^{n-1}\partial^{n-2}f & 0 \\
\partial_y^{n-1}f & \cdots & \partial_x^{n-2}\partial_y^{n-1}f & \partial_x^{n-1}\partial_y^{n-1}f & 1 \\
\partial_y^{n}f & \cdots & \partial_x^{n-2}\partial_y^{n}f & \partial_x^{n-1}\partial_y^{n}f & 0
\end{vmatrix}
= -
\begin{vmatrix}
f & \cdots & \partial_x^{n-2}f & \partial_x^{n-1} \\
\vdots & \cdots & \vdots & \vdots \\
\partial_y^{n-2}f & \cdots & \partial_x^{n-2}\partial_y^{n-2}f & \partial_x^{n-1}\partial^{n-2} \\
\partial_y^{n}f & \cdots & \partial_x^{n-2}\partial_y^{n}f & \partial_x^{n-1}\partial_y^{n}f
\end{vmatrix}
$$

$$
= -\partial_y\tau_n
$$

Molecule solution (4)

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$$|\,\mathbf{0},\cdots,\boldsymbol{n-2},\boldsymbol{n},\boldsymbol{\phi_2}\,| = \begin{vmatrix} f & \cdots & \partial_x^{n-2}f & \partial_x^n & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ \partial_y^{n-1}f & \cdots & \partial_x^{n-2}\partial_y^{n-1}f & \partial_x^n\partial_y^{n-1}f & 0 \\ \partial_y^n f & \cdots & \partial_x^{n-2}\partial_y^n f & \partial_x^n\partial_y^n f & 1 \end{vmatrix} = \begin{vmatrix} f & \cdots & \partial_x^{n-2}f & \partial_x^n \\ \vdots & \cdots & \vdots & \vdots \\ \partial_y^{n-1}f & \cdots & \partial_x^{n-2}\partial_y^{n-1}f & \partial_x^n\partial_y^{n-1}f \end{vmatrix}$$

$$= \partial_x\tau_n$$

$$|\,\mathbf{0},\cdots,\boldsymbol{n-2},\boldsymbol{n},\boldsymbol{\phi_1}\,| = \begin{vmatrix} f & \cdots & \partial_x^{n-2}f & \partial_x^n & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ \partial_y^{n-2}f & \cdots & \partial_x^{n-2}\partial_y^{n-2}f & \partial_x^{n-1}\partial^{n-2}f & 0 \\ \partial_y^{n-1}f & \cdots & \partial_x^{n-2}\partial_y^{n-1}f & \partial_x^n\partial_y^{n-1}f & 1 \\ \partial_y^n f & \cdots & \partial_x^{n-2}\partial_y^n f & \partial_x^n\partial_y^n f & 0 \end{vmatrix} = -\begin{vmatrix} f & \cdots & \partial_x^{n-2}f & \partial_x^n \\ \vdots & \cdots & \vdots & \vdots \\ \partial_y^{n-2}f & \cdots & \partial_x^{n-2}\partial_y^{n-2}f & \partial_x^{n-1}\partial^{n-2}f \\ \partial_y^n f & \cdots & \partial_x^{n-2}\partial_y^n f & \partial_x^n\partial_y^n \end{vmatrix}$$

$$= -\partial_x\partial_y\tau_n$$

Molecule solution (5)

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Plücker Relation

$$
\begin{aligned}
0 = & \ |\,0,\cdots,n-2,n-1,n\,| \times |\,0,\cdots,n-2,\phi_1,\phi_2\,| \\
- & \ |\,0,\cdots,n-2,n-1,\phi_1\,| \times |\,0,\cdots,n-2,n,\phi_2\,| \\
+ & \ |\,0,\cdots,n-2,n-1,\phi_2\,| \times |\,0,\cdots,n-2,n,\phi_1\,| \\
= & \ \tau_{n+1} \times \tau_{n-1} \\
- & \ (-\partial_y\tau_n) \times \partial_x\tau_n \\
& \ \tau_n \times (-\partial_x\partial_y\tau_n)
\end{aligned}
$$

$$\Longrightarrow \quad \left(\partial_x\partial_y\tau_n\right)\tau_n - (\partial_x\tau_n)\left(\partial_y\tau_n\right) = \tau_{n+1}\tau_{n-1} \qquad \text{2DTL}$$

Remark： **Molecular solution is on semi-infinite lattice, but it is possible to restrict it to finite lattice.**

$$f(x,y) = \sum_{i=1}^{N} X_i(x)Y_i(y) \quad \rightarrow \quad \tau_N = \begin{vmatrix} Y_1 & \cdots & Y_N \\ \partial_y Y_1 & \cdots & \partial_y Y_N \\ \vdots & \cdots & \vdots \\ \partial_y^{N-1} Y_1 & \cdots & \partial_y^{N-1} Y_N \end{vmatrix} \times \begin{vmatrix} X_1 & \partial_x X_1 & \cdots & \partial_x^{N-1} X_1 \\ \vdots & \vdots & \cdots & \vdots \\ X_N & \partial_x X_N & \cdots & \partial_x^{N-1} X_N \end{vmatrix} = Y(y) \times X(x)$$

$\Longrightarrow \quad \tau_{N+1} = 0 \quad \because$ **by 2DTL, or “Binet-Cauchy’s Formula”**

Molecule solution (6)

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2DTL and projective differential geometry:

$$\frac{\partial^2 \log h_n}{\partial x \partial y} = h_{n+1} + h_{n-1} - 2h_n, \qquad h_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

describes the transformation of surfaces in the real projective space which admits coordinate systems "conjugate net"(Darboux, 1889). Darboux also constructed the molecule solution on the semi-infinite lattice.

Molecule solution to the Toda lattice:

$$\tau_n = \begin{vmatrix} f(t) & \cdots & \partial_t^{n-1} f(t) \\ \vdots & \vdots & \vdots \\ \partial_t^{n-1} f(t) & \cdots & \partial_t^{2n-2} f(t) \end{vmatrix} \longrightarrow \left(\partial_t^2 \tau_n\right)\tau_n - (\partial_t \tau_n)^2 = \tau_{n+1}\tau_{n-1}, \quad \tau_{-1} = 0, \quad \tau_0 = 1, \quad \tau_1 = f(t)$$

Finite lattice： **Put** $f(t) = \sum_{i=1}^{N} e^{\lambda_i t + \mu_i}$

short-circuit: $V_n \to 0$

↓

$$\begin{cases} \dfrac{d}{dt}\log V_n = I_n - I_{n+1}, \\ \dfrac{dI_n}{dt} = V_{n-1} - V_n \end{cases} \qquad \begin{cases} V_n = \dfrac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} \\ I_n = \dfrac{d}{dt}\log\dfrac{\tau_{n-1}}{\tau_n} \end{cases} \qquad V_0 = 0, \quad V_N = 0$$

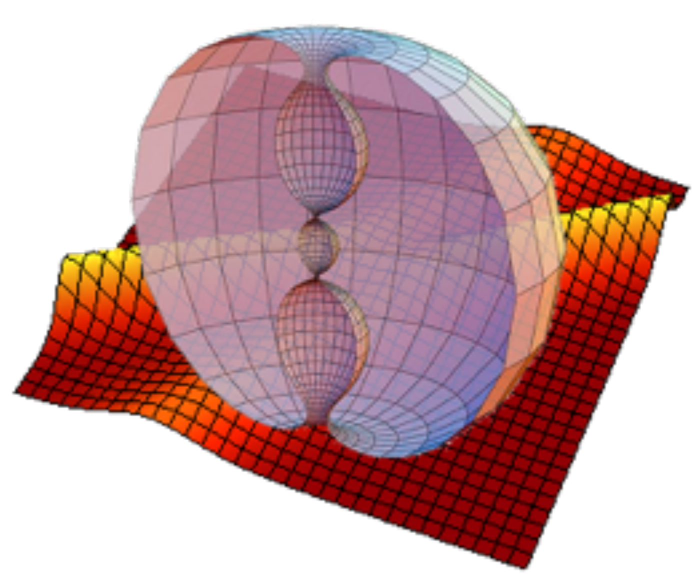
L matrix in Lax formalism
→ diagonal matrix
→ numerical scheme of eigenvalues

2：Reductions

Contents and Keywords

- **Reduction**

  A procedure to yield a new equation by restricting the solution space.
- **Sine-Gordon equation, Toda lattice equation**
- **Reduction on the level of solution**



Reduction

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- **Reduction:** Procedure to yield a new equation by restricting the solution space (parameters of solutions)

- **2DTL → 1DTL:** Put t = x+y, s = x-y and impose $\frac{\partial q_n}{\partial s} = 0$

$$\frac{\partial^2 q_n}{\partial x \partial y} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \to \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) q_n = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \to \boxed{\frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}}$$

- **2DTL → sinh-Gordon:** impose 2-periodicity $q_{n+2} = q_n$

$$\frac{\partial^2 q_n}{\partial x \partial y} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}} \to \begin{cases} \frac{\partial^2 q_0}{\partial x \partial y} = e^{q_1 - q_0} - e^{q_0 - q_1} \\ \frac{\partial^2 q_1}{\partial x \partial y} = e^{q_0 - q_1} - e^{q_1 - q_0} \end{cases} \to \frac{\partial^2 v}{\partial x \partial y} = 2\left(e^{-v} - e^{v}\right), \quad v := q_0 - q_1$$

$$\longrightarrow \boxed{\frac{\partial^2 v}{\partial x \partial y} = -4 \sinh v} \quad \text{sinh-Gordon equation}$$

$$\boxed{\frac{\partial^2 \theta}{\partial x \partial y} = -4 \sin \theta} \quad \text{sine-Gordon equation} \quad v = \sqrt{-1}\theta \in \sqrt{-1}\mathbb{R}$$

Reduction to Toda Lattice (I)

- **2DTL → 1DTL:** Impose restriction on τ function to realize $\dfrac{\partial q_n}{\partial s} = 0$

$$q_n = \log \frac{\tau_{n-1}}{\tau_n} \quad \Longrightarrow \quad \partial_s q_n = \frac{\partial_s \tau_{n-1}}{\tau_{n-1}} - \frac{\partial_s \tau_n}{\tau_n} = 0 \quad \Longrightarrow \quad \boxed{\partial_s \tau_n = \text{const.} \times \tau_n}$$

Soliton solution: $\tau_n = \det\left(\varphi^{(i)}_{n+j-1}\right)_{i,j=1,\ldots,N}$

$$\varphi^{(i)}_n = p_i^n e^{p_i x - \frac{y}{p_i}} + q_i^n e^{q_i x - \frac{y}{q_i}} = p_i^n \exp\left[\frac{1}{2}\left(p_i - \frac{1}{p_i}\right)t + \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)s\right] + q_i^n \exp\left[\frac{1}{2}\left(q_i - \frac{1}{q_i}\right)t + \frac{1}{2}\left(q_i + \frac{1}{q_i}\right)s\right]$$

$$\to \partial_s \varphi^{(i)}_n = \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)p_i^n \exp\left[\frac{1}{2}\left(p_i - \frac{1}{p_i}\right)t + \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)s\right] + \frac{1}{2}\left(q_i + \frac{1}{q_i}\right)q_i^n \exp\left[\frac{1}{2}\left(q_i - \frac{1}{q_i}\right)t + \frac{1}{2}\left(q_i + \frac{1}{q_i}\right)s\right] \propto \varphi^{(i)}_n$$

$$\to p_i + \frac{1}{p_i} = q_i + \frac{1}{q_i} \quad \longrightarrow \quad (p_i - q_i)\left(1 - \frac{1}{p_i q_i}\right) = 0 \quad \longrightarrow \quad \boxed{q_i = \frac{1}{p_i}}, \quad \partial_s \varphi^{(i)}_n = \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)\varphi^{(i)}_n$$

$$\Longrightarrow \partial_s \tau_n = C_N\, \tau_n, \quad C_N = \sum_{i=1}^{N} \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)$$

Reduction to Toda Lattice (2)

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$$t = x + y, \quad s = x - y, \quad q_i = \frac{1}{p_i}, \quad \partial_s \tau_n = C_N\, \tau_n, \quad C_N = \sum_{i=1}^{N} \frac{1}{2}\left(p_i + \frac{1}{p_i}\right)$$

2DTL $\left(\partial_x \partial_y \tau_n\right)\tau_n - \left(\partial_x \tau_n\right)\left(\partial_y \tau_n\right) = \tau_{n+1}\tau_{n-1} - \tau_n^2$

$$\mathrm{LHS} = \left(\partial_t^2 - \partial_s^2\right)\tau_n \times \tau_n - \left(\partial_t - \partial_s\right)\tau_n \times \left(\partial_t + \partial_s\right)\tau_n = \left(\partial_t^2 - C_N^2\right)\tau_n \times \tau_n - \left(\partial_t - C_N\right)\tau_n \times \left(\partial_t + C_N\right)\tau_n$$

$$= \left(\partial_t^2 \tau_n\right)\tau_n - \left(\partial_t \tau_n\right)^2 \quad \Longrightarrow \quad \boxed{\left(\partial_t^2 \tau_n\right)\tau_n - \left(\partial_t \tau_n\right)^2 = \tau_{n+1}\tau_{n-1} - \tau_n^2}$$

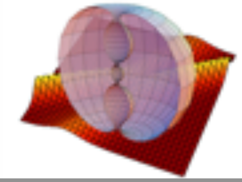
1DTL!

- **Bilinear equation and τ function for 1DTL:**

$$\frac{1}{2} D_t^2\, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2$$

$$\tau_n = \begin{vmatrix} \varphi_n^{(1)} & \varphi_{n+1}^{(1)} & \cdots & \varphi_{n+N-1}^{(1)} \\ \varphi_n^{(2)} & \varphi_{n+1}^{(2)} & \cdots & \varphi_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_n^{(N)} & \varphi_{n+1}^{(N)} & \cdots & \varphi_{n+N-1}^{(N)} \end{vmatrix} \qquad \varphi_n^{(i)} = p_i^n\, e^{\frac{t}{2}\left(p_i - \frac{1}{p_i}\right) + \eta_{i0}} + p_i^{-n}\, e^{-\frac{t}{2}\left(p_i - \frac{1}{p_i}\right) + \xi_{i0}}$$

Reduction to Sinh-Gordon(I)



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- **2DTL → Sinh-Gordon:** Impose restriction on τ function to realize $q_{n+2} = q_n$

$$q_n = \log \frac{\tau_{n-1}}{\tau_n} \quad \Longrightarrow \quad \log \frac{\tau_{n+1}}{\tau_{n+2}} = \log \frac{\tau_{n-1}}{\tau_n} \quad \Longrightarrow \quad \boxed{\tau_{n+2} = \text{const.} \times \tau_n}$$

- **Soliton Solution:** $\tau_n = \det\left(f^{(i)}_{n+j-1}\right)_{i,j=1,\ldots,N}$

$$f^{(k)}_n = p_k^n e^{p_k x - \frac{y}{p_k}} + q_k^n e^{q_k x - \frac{y}{q_k}} \rightarrow \quad f^{(k)}_{n+2} = p_k^2\, p_k^n e^{p_k x - \frac{y}{p_k}} + q_k^2\, q_k^n e^{q_k x - \frac{y}{q_k}} \propto f^{(k)}_n \rightarrow \quad p_k^2 = q_k^2$$

$$\longrightarrow \quad \boxed{q_k = -p_k} \quad \longrightarrow \quad f^{(k)}_{n+2} = p_k^2\, f^{(k)}_n \quad \Longrightarrow \quad \tau_{n+2} = \lambda\, \tau_n, \quad \lambda = \prod_{i=1}^{N} p_i^2$$

- **Bilinear equation:** $v = q_0 - q_1 = \log \dfrac{\tau_{-1}\tau_1}{\tau_0^2} = \log \dfrac{\tau_1^2}{\tau_0^2} - \log \lambda$

2DTL $\quad \dfrac{1}{2} D_x D_y\, \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2 \quad \Longrightarrow \quad \begin{cases} \dfrac{1}{2} D_x D_y\, \tau_0 \cdot \tau_0 = \dfrac{1}{\lambda}\tau_1^2 - \tau_0^2 \\ \dfrac{1}{2} D_x D_y\, \tau_1 \cdot \tau_1 = \lambda \tau_0^2 - \tau_1^2 \end{cases}$ **sinh-Gordon**

Reduction to Sine-Gordon

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Bilinear equation and Casorati determinant solution to Sinh-Gordon equation

$$v_{xy} = -4\sinh v, \qquad v = 2\log\frac{\tau_1}{\tau_0} - \log\lambda$$

$$\begin{cases} \frac{1}{2}D_xD_y\,\tau_0\cdot\tau_0 = \frac{1}{\lambda}\,\tau_1^2 - \tau_0^2 \\ \frac{1}{2}D_xD_y\,\tau_1\cdot\tau_1 = \lambda\,\tau_0^2 - \tau_1^2 \end{cases} \qquad \tau_n = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix},$$

$$f_n^{(k)} = p_k^n e^{p_kx-\frac{y}{p_k}+\eta_{k0}} + (-p_k)^n e^{-\left(p_kx-\frac{y}{p_k}\right)+\xi_{k0}}, \qquad \lambda = \prod_{i=1}^{N} p_i^2$$

Reduction to Sine-Gordon equation

sinh-Gordon 方程式 sine-Gordon 方程式

$$\boxed{\frac{\partial^2 v}{\partial x\partial y} = -4\sinh v} \longrightarrow \quad v = i\theta \in i\mathbb{R} \quad \longrightarrow \boxed{\frac{\partial^2 \theta}{\partial x\partial y} = -4\sin\theta}$$

It is not trivial to realize the restriction v: pure imaginary on the level of τ function

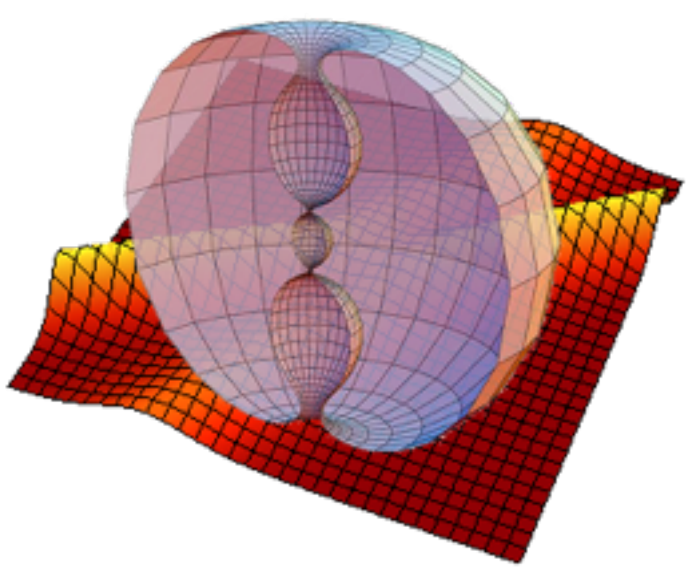
Gram determinan is convenient! (Ohta's lecture)

In Casorati determinant, $\eta_{i0} \in \mathbb{R}, \quad \xi_i = \frac{\pi i}{2}$ **is sufficient (proof is a bit complicated)**

3：Bäcklund Transformation

Contents and Keywords

- BT from calculus of bilinear equations
- Lax Formalism from BT
- New solutions from BT



Bäcklund Transformation (I)

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Theorem（BT of 1DTL）：Let τ_n be a solution to 1DTL

$$\frac{1}{2}D_t^2\,\tau_n\cdot\tau_n=\tau_{n+1}\tau_{n-1}-\tau_n^2 \qquad (\bigstar)$$

For parameters λ_1, λ_2, λ_3, if $\overline{\tau}_n$ satisfies

$$\begin{aligned} D_t\,\tau_n\cdot\overline{\tau}_n &= \lambda_1\tau_{n+1}\overline{\tau}_{n-1}-\lambda_2\tau_n\overline{\tau}_n \\ D_t\,\tau_{n+1}\cdot\overline{\tau}_n &= -\frac{1}{\lambda_1}\tau_n\overline{\tau}_{n+1}+\lambda_3\tau_{n+1}\overline{\tau}_n \end{aligned} \qquad (\star)$$

then $\overline{\tau}_n$ also satisfies（★）. Conversely, if $\overline{\tau}_n$ satisfy（★），and τ_n satisfies（☆），then τ_n is also a solution of（★）.

Proof:

$$P=\left[\frac{1}{2}D_xD_y\,\tau_n\cdot\tau_n-\tau_{n+1}\tau_{n-1}+\tau_n^2\right](\overline{\tau}_n)^2-(\tau_n)^2\left[\frac{1}{2}D_xD_y\,\overline{\tau}_n\cdot\overline{\tau}_n-\overline{\tau}_{n+1}\overline{\tau}_{n-1}+(\overline{\tau}_n)^2\right]$$

It suffices to show P=0 if τ_n satisfies（☆）and $\overline{\tau}_n$ satisfies（★）.

Bäcklund Transformation (2)

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Key： "Exchange formula" of Hirota Derivative

Prop： For arbitrary $x, y, \tau_n, \overline{\tau}_n$ the following formulas hold:

(1) $$\left[D_xD_y\, \tau_n\cdot\tau_n\right]\overline{\tau}_n^2-(\tau_n)^2\left[D_xD_y\,\overline{\tau}_n\cdot\overline{\tau}_n\right]=2D_x\left(D_y\,\tau_n\cdot\overline{\tau}_n\right)\cdot\overline{\tau}_n\tau_n,$$

(2) $$D_x\,(\tau_{n+1}\overline{\tau}_{n-1})\cdot(\overline{\tau}_n\tau_n)=\left[D_x\,\tau_{n+1}\cdot\overline{\tau}_n\right]\overline{\tau}_{n-1}\tau_n+\tau_{n+1}\overline{\tau}_n\left[D_x\,\overline{\tau}_{n-1}\cdot\tau_n\right]$$

Verified directly. Refer to Hirota's book for "elegant" method to generate similar formulas.

$$P=\left[\left[\frac{1}{2}D_t^2\,\tau_n\cdot\tau_n\right]-\tau_{n+1}\tau_{n-1}+\tau_n^2\right](\overline{\tau}_n)^2-(\tau_n)^2\left[\left[\frac{1}{2}D_t^2\,\overline{\tau}_n\cdot\overline{\tau}_n\right]-\overline{\tau}_{n+1}\overline{\tau}_{n-1}+(\overline{\tau}_n)^2\right]$$

$$=D_t\left[D_t\,\tau_n\cdot\overline{\tau}_n\right]\cdot\overline{\tau}_n\tau_n+\tau_n\overline{\tau}_{n+1}\overline{\tau}_{n-1}\tau_n+\tau_{n-1}\overline{\tau}_n\overline{\tau}_n\tau_{n+1}$$

$\because$ exchange formula (1)

$$=D_t\left[D_t\,\tau_n\cdot\overline{\tau}_n-\lambda_1\tau_{n+1}\overline{\tau}_{n-1}\right]\cdot\overline{\tau}_n\tau_n+\lambda_1\left[\left[D_x\,\overline{\tau}_{n+1}\cdot\tau_n\right]+\lambda_1^{-1}\overline{\tau}_n\tau_{n+1}\right]\tau_{n-1}\overline{\tau}_n+\lambda_1\left[\left[D_x\,\tau_n\cdot\overline{\tau}_{n-1}\right]+\lambda_1^{-1}\tau_{n-1}\overline{\tau}_n\right]\overline{\tau}_n\tau_{n+1}$$

$\because$ exchange formula (2)

Therefore, if

$$D_t\,\tau_n\cdot\overline{\tau}_n-\lambda_1\tau_{n+1}\overline{\tau}_{n-1}=-\lambda_2\tau_n\overline{\tau}_n \quad\longleftarrow\quad \text{1st term: } D_t\,f\cdot f=0$$

$$D_t\,\tau_{n+1}\cdot\overline{\tau}_n=-\frac{1}{\lambda_1}\tau_n\overline{\tau}_{n+1}+\lambda_3\tau_{n+1}\overline{\tau}_n \quad\longleftarrow\quad \text{2nd,3rd term: } \lambda_3 \text{ terms cancel}$$

then P becomes 0. q.e.d.

Bäcklund Transformation (3)

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$$q_n = \log\frac{\tau_{n-1}}{\tau_n}, \quad \overline{q}_n = \log\frac{\overline{\tau}_{n-1}}{\overline{\tau}_n}, \qquad \textbf{1DTL:} \quad \frac{d^2 q_n}{dt^2} = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}$$

BT： **note that** $\dfrac{D_t\, f\cdot g}{fg} = \dfrac{f'g - fg'}{fg} = \dfrac{f'}{f} - \dfrac{g'}{g} = (\log f)' - (\log g)'$

$$D_t\, \tau_n \cdot \overline{\tau}_n = \lambda_1 \tau_{n+1}\overline{\tau}_{n-1} - \lambda_2 \tau_n \overline{\tau}_n \qquad (\log\tau_n)' - (\log\overline{\tau}_n)' = \lambda_1 \frac{\tau_{n+1}\overline{\tau}_{n-1}}{\tau_n\overline{\tau}_n} - \lambda_2$$

$$\longrightarrow$$

$$D_t\, \tau_{n+1} \cdot \overline{\tau}_n = -\frac{1}{\lambda_1}\tau_n\overline{\tau}_{n+1} + \lambda_3 \tau_{n+1}\overline{\tau}_n \qquad (\log\tau_{n+1})' - (\log\overline{\tau}_n)' = -\frac{1}{\lambda_1}\frac{\tau_n\overline{\tau}_{n+1}}{\tau_{n+1}\overline{\tau}_n} + \lambda_3$$

(upper) $_{n-1}$ **- (lower)** $_{n-1}$ **:**

$$\left(\log\frac{\tau_{n-1}}{\tau_n}\right)' = \lambda_1 \frac{\tau_n\overline{\tau}_{n-2}}{\tau_{n-1}\overline{\tau}_{n-1}} + \frac{1}{\lambda_1}\frac{\tau_{n-1}\overline{\tau}_n}{\tau_n\overline{\tau}_{n-1}} - \lambda_2 + \lambda_3 \longrightarrow \boxed{\frac{dq_n}{dt} = \lambda_1 e^{-q_n + \overline{q}_{n-1}} + \frac{1}{\lambda_1}e^{q_n - \overline{q}_n} - \lambda_2 + \lambda_3}$$

(upper) $_{n}$ **- (lower)** $_{n-1}$ **:**

$$\left(\log\frac{\overline{\tau}_{n-1}}{\overline{\tau}_n}\right)' = \lambda_1 \frac{\tau_{n+1}\overline{\tau}_{n-1}}{\tau_n\overline{\tau}_n} + \frac{1}{\lambda_1}\frac{\tau_{n-1}\overline{\tau}_n}{\tau_n\overline{\tau}_{n-1}} - \lambda_2 + \lambda_3 \longrightarrow \boxed{\frac{d\overline{q}_n}{dt} = \lambda_1 e^{-q_{n+1} + \overline{q}_n} + \frac{1}{\lambda_1}e^{q_n - \overline{q}_n} - \lambda_2 + \lambda_3}$$

Same as the BT shown in Chapter 1!

Bäcklund Transformation (4)

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BT yields Lax form: **Put** $\overline{\tau}_n = \tau_n \Psi_{n+1}$ **!**

$$D_t\, \tau_n \cdot \overline{\tau}_n = \lambda_1 \tau_{n+1}\overline{\tau}_{n-1} - \lambda_2 \tau_n \overline{\tau}_n$$
$$D_t\, \tau_{n+1} \cdot \overline{\tau}_n = -\frac{1}{\lambda_1}\tau_n \overline{\tau}_{n+1} + \lambda_3 \tau_{n+1}\overline{\tau}_n$$

$$\longrightarrow$$

$$\tau_n'(\tau_n \Psi_{n+1}) - \tau_n(\Psi_{n+1}\tau_n)' = \lambda_1 \tau_{n+1}\tau_{n-1}\Psi_n - \lambda_2 \tau_n^2 \Psi_{n+1}$$
$$\tau_{n+1}'(\Psi_{n+1}\tau_n) - \tau_{n+1}(\Psi_{n+1}\tau_n)' = -\frac{1}{\lambda_1}\tau_n \tau_{n+1}\Psi_{n+2} + \lambda_3 \tau_{n+1}\tau_n \Psi_{n+1}$$

$$\longrightarrow \quad -\Psi_{n+1}' = \lambda_1 \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}\Psi_n - \lambda_2 \Psi_{n+1}$$
$$-\Psi_{n+1}' + \left(\log \frac{\tau_{n+1}}{\tau_n}\right)' \Psi_{n+1} = -\frac{1}{\lambda_1}\Psi_{n+2} + \lambda_3 \Psi_{n+1}$$

$$\longrightarrow \quad \Psi_n' = -\lambda_1(1 + V_{n-1})\Psi_{n-1} + \lambda_2 \Psi_n$$
$$\Psi_n' = -(I_n + \lambda_3)\Psi_n + \frac{1}{\lambda_1}\Psi_{n+1}$$

$$\longrightarrow \quad \lambda_1(1 + V_{n-1})\Psi_{n-1} - I_n \Psi_n + \frac{1}{\lambda_1}\Psi_{n+1} = (\lambda_2 + \lambda_3)\Psi_n$$
$$\Psi_n' = -\lambda_1(1 + V_{n-1})\Psi_{n-1} + \lambda_2 \Psi_n$$

$$\longrightarrow \quad \boxed{\begin{aligned} &(1 + V_{n-1})\Psi_{n-1} + I_n \Psi_n + \Psi_{n+1} = \lambda \Psi_n \\ &\Psi_n' = (1 + V_{n-1})\Psi_{n-1} \end{aligned}}$$

Lax Form

where $\lambda_1 = -1, \quad \lambda_2 = 0, \quad \lambda_3 = -\lambda.$

or $(-\lambda_1)^n e^{\lambda_2 t}\Psi_n \to \Psi_n, \quad \lambda_3 = -\lambda.$

Bäcklund Transformation (5)

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Theorem（BT of 2DTL）： Let τ_n be a solution to 2DTL

$$\frac{1}{2}D_xD_y\,\tau_n\cdot\tau_n=\tau_{n+1}\tau_{n-1}-\tau_n^2 \qquad (\bigstar)$$

For parameters λ_1, λ_2, λ_3, if $\overline{\tau}_n$ satisfies

$$\begin{aligned} D_y\,\tau_n\cdot\overline{\tau}_n &= \lambda_1\tau_{n+1}\overline{\tau}_{n-1}-\lambda_2\tau_n\overline{\tau}_n \\ D_x\,\tau_{n+1}\cdot\overline{\tau}_n &= -\frac{1}{\lambda_1}\tau_n\overline{\tau}_{n+1}+\lambda_3\tau_{n+1}\overline{\tau}_n \end{aligned} \qquad (\star)$$

then $\overline{\tau}_n$ also satisfies（★）. Conversely, if $\overline{\tau}_n$ satisfy（★）, and τ_n satisfies（☆）, then τ_n is also a solution of（★）.

問：Derive the above Theorem by the similar discussion with 1DTL case

Consider Casorati determinant solution as τ_n：

$$\tau_n(N)=\begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix} \qquad \begin{aligned} \partial_x f_n^{(k)} &= f_{n+1}^{(k)} \\ \partial_y f_n^{(k)} &= -f_{n-1}^{(k)} \end{aligned} \quad \longrightarrow \quad \overline{\tau}_n=\tau_n(N+1)$$

Bäcklund Transformation (6)

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Bilinear eq. of 2DTL was derived by applying Laplace expansion to:

$$0 = \begin{vmatrix} -1 & 0 \;\; 1 \;\; \cdots \;\; N-2 & \emptyset & N-1 & N \\ -1 & \emptyset & 1 \;\; \cdots \;\; N-2 & N-1 & N \end{vmatrix}$$

modify the orange part:

$$0 = \begin{vmatrix} -1 & 0 \;\; 1 \;\; \cdots \;\; N-2 & \emptyset & N-1 & \phi_2 \\ -1 & \emptyset & 1 \;\; \cdots \;\; N-2 & N-1 & \phi_2 \end{vmatrix} \qquad \phi_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Laplace expansion:

$$\begin{aligned} 0 = \; & |-1, 0, 1, \cdots, N-2| \times |1, \cdots, N-2, N-1, \phi_2| \\ + \; & |0, 1, \cdots, N-2, N-1| \times |-1, 1, \cdots, N-2, \phi_2| \\ - \; & |0, 1, \cdots, N-2, \phi_2| \times |-1, 1, \cdots, N-2, N-1| \end{aligned}$$

$$\Longrightarrow \quad 0 = \tau_{n-1}(N) \times \tau_{n+1}(N-1) + \tau_n(N) \times \left(-\partial_y \tau_n(N-1)\right) - \tau_n(N-1) \times (-\partial_y \tau_n(N))$$

$$\Longrightarrow \quad \boxed{D_y\, \tau_n(N) \cdot \tau_n(N+1) = \tau_{n+1}(N)\tau_{n-1}(N+1)} \qquad \tau_n = \tau_n(N),\ \overline{\tau}_n = \tau_n(N+1),\ \lambda_1 = 1,\ \lambda_2 = 0$$

Bäcklund Transformation (7)

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問： Let $\tau_n = \tau_n(N),\ \overline{\tau}_n = \tau_n(N+1)$ **Derive another BT:**

$$D_x\, \tau_{n+1}\cdot\overline{\tau}_n = -\frac{1}{\lambda_1}\tau_n\overline{\tau}_{n+1} + \lambda_3\tau_{n+1}\overline{\tau}_n$$

Answer： Apply Laplace expantion to:

$$0 = \left|\begin{array}{cccc:ccc:ccc} \mathbf{0} & \mathbf{1} & \cdots & \boldsymbol{N-2} & & \emptyset & & \boldsymbol{N-1} & \boldsymbol{N} & \boldsymbol{\phi_2} \\ \hdashline & & \emptyset & & \mathbf{1} & \cdots & \boldsymbol{N-2} & \boldsymbol{N-1} & \boldsymbol{N} & \boldsymbol{\phi_2} \end{array}\right|$$

$$\begin{aligned} 0 = \ & |\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1}| \times |\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N},\boldsymbol{\phi_2}| \\ - \ & |\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N}| \times |\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1},\boldsymbol{\phi_2}| \\ + \ & |\mathbf{0},\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{\phi_2}| \times |\mathbf{1},\cdots,\boldsymbol{N-2},\boldsymbol{N-1},\boldsymbol{N}| \end{aligned}$$

$$\Longrightarrow \quad 0 = \tau_n(N)\times\partial_x\tau_{n+1}(N-1) - \partial_x\tau_n(N)\times\tau_{n+1}(N-1) + \tau_n(N-1)\times\tau_{n+1}(N)$$

$$\Longrightarrow \quad \boxed{D_x\,\tau_{n+1}(N)\cdot\tau_n(N+1) = -\tau_n(N)\tau_{n-1}(N+1)} \quad \tau_n = \tau_n(N),\ \overline{\tau}_n = \tau_n(N+1),\ \lambda_1 = 1,\ \lambda_3 = 0$$

Bäcklund Transformation (7)

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- **Lax form of 2DTL:** $\overline{\tau}_n = \tau_n \Psi_{n+1}$

$$\begin{cases} \partial_y \Psi_n = -\lambda_1 (1 + V_{n-1}) \Psi_{n-1} + \lambda_2 \Psi_n \\ \partial_x \Psi_n = -(I_n + \lambda_3) \Psi_n + \dfrac{1}{\lambda_1} \Psi_{n+1} \end{cases}$$

両立条件 $\partial_x(\partial_y \Psi_n) = \partial_y(\partial_x \Psi_n) \quad \to \quad$ 2DTL

- **Reducion of BT：** the same as reduction of equation
  - 1DTL: t = x+y, s = x-y and kill s dependence
  - Sinh-Gordon, Sine-Gordon: impose 2-periodicity

BT of Sine-Gordon equation (A.V. Bäcklund, 1875)

$$\left(\frac{\theta - \overline{\theta}}{2}\right)_y = -2\lambda_1 \sin \frac{\theta + \overline{\theta}}{2}$$
$$\left(\frac{\theta + \overline{\theta}}{2}\right)_x = -\frac{2}{\lambda_1} \sin \frac{\theta - \overline{\theta}}{2}$$
$$\longrightarrow \quad \theta_{xy} = -4 \sin \theta, \qquad \overline{\theta}_{xy} = -4 \sin \overline{\theta}$$

Transformation of surface with constant negative curvature!